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Local systems of simple locally finite associative algebras*

Research Article

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Abstract: In this paper, we study local systems of locally finite associative algebras over fields of free characteristic. We describe the perfect local systems and study the relation between them and their corresponding locally finite associative algebras. 1-perfect and conical local systems are also be considered and described briefly.

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1. Introduction

Throughout the paper, the field $\mathbb{F}$ is algebraically closed of characteristic $p \geq 0$ and A is an infinite countable-dimensional locally finite associative algebra over $\mathbb{F}$. Recall that an algebra A is called locally finite if every finite set of elements is contained in a finite dimensional subalgebra of A [1]. A is called *locally simple* if for any finite subset U of A, there is a finite dimensional simple subalgebra that contains U. Note that we do not require A to have an identity element.

Locally finite Lie algebras were described by Bahturin and Strade [3] in 1995. They described these algebras in terms of local systems of locally finite Lie algebras. Recall that a system of finite dimensional subalgebras $\{A_\alpha\}_{\alpha\in\Gamma}$ of an algebra A over $\mathbb{F}$ is said to be a *local system for* A if $A = \cup_{\alpha\in\Gamma} A_\alpha$ and for each $\alpha, \beta \in \Gamma$, there is $\gamma \in \Gamma$ such that $A_\alpha, A_\beta \subseteq A_\gamma$. Bahturin and Strade in [2] provided some examples to describe locally finite Lie algebras. In several papers (see for example [4], [5], [7] and [11]) Baranov et. al. classified simple locally finite Lie algebras over algebraically closed fields of characteristic zeros. They showed that there are two classes of locally finite Lie algebras which have can be characterised in many different ways. These are the simple diagonal locally finite Lie algebras and the finitary simple Lie

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algebras. Inner ideals of the finitary simple Lie algebras were studied by Fernandez Lopes, Garcia and Gomez Lozano in [12], while inner ideals of the other class were studied by Baranov and Rowley in [9].

In 2004, Bahturin, Baranov and Zalesski [1] studied the simple locally finite associative algebras over algebraically closed fields of zero characteristic. They highlighted the relation between them and locally finite Lie algebras over algebraically closed field of characteristic 0. They proved that simple Lie subalgebras of locally finite associative ones are either finite dimensional or isomorphic to the Lie algebra of skew symmetric elements of some (Type1) involution simple locally finite associative algebras.

Baranov in [6] proved that there is a natural bijective correspondence between such Lie algebras and locally involution simple associative algebras over algebraically closed fields of any characteristic $\neq 2$. Thus, to classify locally finite Lie algebras, we need to study locally finite associative algebras briefly. This requires a good understanding of their local systems.

In this paper, we study local systems of locally finite associative algebras over fields of characteristic $p \geq 0$. Some of the results in this papers were mentioned in [1] in the case when $p = 0$. We start with some of the preliminaries in section two. Section three is devoted to the study of local systems of locally finite associative algebras.

In section four, we describe the perfect local systems and study the relation between them and their corresponding locally finite associative algebras. Finally, the 1-perfect and conical local systems were described.

2. Preliminaries

Recall that $\mathbb{F}$ is an algebraically closed field of characteristic $p \geq 0$.

Definition 2.1. *[1] A locally finite algebra is an algebra A over a field $\mathbb{F}$ in which every finite set of elements of A is contained in a finite dimensional subalgebra of A.*

As an example of a locally finite associative algebra is the algebra $M_\infty(\mathbb{F})$ of infinite matrices with finite numbers of non-zero entries, that is,

$$M_\infty(\mathbb{F}) = \cup_{n=1}^{\infty} M_n(\mathbb{F}), \tag{1}$$

where the algebra $M_n(\mathbb{F})$ can be embedded in $M_{(n+1)}(\mathbb{F})$ by putting $M_n(\mathbb{F})$ in the left upper hand corner and bordering the last column and row by 0's.

Definition 2.2. *Let A be an algebra over a field $\mathbb{F}$. A system of finite dimensional subalgebras $\{A_\alpha\}_{\alpha\in\Gamma}$ of A is said to be a* local system *for A if $A = \cup_{\alpha\in\Gamma} A_\alpha$ and for each $\alpha, \beta \in \Gamma$, there is $\gamma \in \Gamma$ such that $A_\alpha, A_\beta \subseteq A_\gamma$.*

Put $\alpha \leq \beta$ if $A_\alpha \subseteq A_\beta$. Then Γ is a directed set and

$$A = \varinjlim A_\alpha \tag{2}$$

is a direct limit of the algebras A_α. Recall that A is called perfect algebra if $AA = A$.

Definition 2.3. *1. A local system is said to be* perfect *(resp.* simple, semisimple, nilpotent, ... etc*) if it consists of perfect (simple, semisimple, nilpotent, ... etc) algebras.*

2. A locally finite algebra A is called locally perfect *(resp.* simple, semisimple, nilpotent, ... etc*) if it consists of a perfect (simple, semisimple, nilpotent, ... etc) local system.*

Example 2.4. *Suppose that A is locally simple. Then there is a chain of simple subalgebras*

$$A_1 \subset A_2 \subset A_3 \subset \dots$$

of A such that $A = \cup_{i=1}^{\infty} A_i$. We can view A as the direct limit $\lim_{\longleftarrow} A_i$ for the sequence

$$A_1 \to A_2 \to A_3 \to \dots$$

of injective homomorphisms of finite dimensional simple associative algebras A_i. Since $\mathbb{F}$ is algebraically closed, each A_i can be identified with the algebra $M_{n_i}(\mathbb{F})$ of all $n_i \times n_i$-matrices over $\mathbb{F}$ for some n_i. Moreover, we may assume that each embedding $A_i \to A_{i+1}$ can be written in the following matrix form

$$M \to diag(M, \dots, M, 0, \dots, 0), \quad M \in M_{n_i}(\mathbb{F}).$$

3. Local systems of locally finite associative algebras

Lemma 3.1. *Let A be a locally finite associative algebra over $\mathbb{F}$ and let $\{A_\alpha\}_{\alpha\in\Gamma}$ be a system of finitely generated subalgebras of A. Then $\{A_\alpha\}_{\alpha\in\Gamma}$ is a local system of A if and only if for every finite dimensional subspace P of A there exists $\beta \in \Gamma$ such that $P \subseteq A_\beta$.*

Proof. Suppose first that $\{A_\alpha\}_{\alpha\in\Gamma}$ is a local system of A. Let P be a finitely generated subalgebra of A and let $\{p_1, \dots, p_n\}$ be a basis of P. Then $A = \cup_{\alpha\in\Gamma} A_\alpha$ and for each $1 \le i \le n$, there is $A_i \in \{A_\alpha\}_{\alpha\in\Gamma}$ such that $p_i \in A_i$. Thus $P \subseteq A_\beta$ for some $\beta \in \Gamma$, as required.

Suppose now that for every finite dimensional subspace P of A there exists $\beta \in \Gamma$ such that $P \subseteq A_\beta$. We need to show that $\{A_\alpha\}_{\alpha\in\Gamma}$ is a local system of A. Let $x \in A$. Then for every subspace P_x generated by x, there is $\beta \in \Gamma$ such that $P_x \subseteq A_\beta$, so $A = \cup_{\alpha\in\Gamma} A_\alpha$. It remains to note that for any $\alpha, \beta \in \Gamma$, there is $\gamma \in \Gamma$ such that $A_\alpha, A_\beta \subseteq A_\gamma$. □

Lemma 3.2. *Suppose that $\{A_\alpha\}_{\alpha\in\Gamma}$ is a local system of a locally finite associative algebra A over $\mathbb{F}$. If $\Gamma = \cup_{i=1}^{r} \Gamma_i$, then $\{A_\alpha\}_{\alpha\in\Gamma_k}$ (for some $1 \le k \le r$) is a local system of A.*

Proof. Suppose that $\Gamma = \cup_{i=1}^{r} \Gamma_i$. We may assume that Γ is a is a disjoint union of Γ_i of Γ (because if it is not, then we can decompose it as a disjoint union of subsets). We are going to prove by contradiction that $\{A_\alpha\}_{\alpha\in\Gamma_k}$, for some $1 \le k \le r$, is a local system of A. Assume to the contrary that $\{A_\alpha\}_{\alpha\in\Gamma_i}$ is not a local system of A for each $1 \le i \le r$. Then there is a finite dimensional subspace P_i of A such that $P_i \notin \{A_\alpha\}_{\alpha\in\Gamma_i}$ for each i. Consider the subspace $P = \bigoplus_{i=1}^{r} P_i$ of A. Then P is finite dimensional with $P \nsubseteq A_\alpha$ for all $\alpha \in \Gamma$ (because Γ is a disjoint union of the subsets Γ_i). Thus, $\{A_\alpha\}_{\alpha\in\Gamma}$ is not a local system of A, a contradiction. □

Lemma 3.3. *Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be a local system of A and let P be a finite dimensional subspace of A. Then $\{A_\beta\}_{\beta\in\Gamma_P}$ is a local system of A, where $\Gamma_P = \{\beta \in \Gamma \mid P \subseteq A_\beta\}$.*

Proof. By Lemma 3.1, there is $\beta \in \Gamma$ such that $A_\beta \supset P$. Let P_1 be a finite dimensional subalgebra of A. Then $P_2 = P + P_1$ is a finite dimensional subalgebra of A, so by Lemma 3.1, There is $\gamma \in \Gamma$ such that $A_\gamma \supset P_2$ with $\gamma \ge \beta$. Continuing with this process we get the set $\Gamma_P = \{\beta \in \Gamma \mid A_\beta \supset P\} \subset \Gamma$. By Lemma 3.2, $\{A_\beta\}_{\beta\in\Gamma_P}$ is a local system of A. □

Proposition 3.4. *Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be a local system of a simple locally finite associative algebra A over $\mathbb{F}$. The following hold.*

1. [13] Let $\{I_\alpha\}_{\alpha\in\Gamma}$ be a system of ideals such that I_α is an ideal of A_α for each $\alpha \in \Gamma$. Then either $\cap_{\alpha\in\Gamma} I_\alpha = 0$ or for every $k \in \Gamma$ or there exists $\beta_k \in \Gamma$ such that $A_k \subseteq I_{\beta_k}$.

2. Suppose that $\Gamma_P = \{\beta \in \Gamma \mid P \subseteq A_\beta\}$. Then $\{A_\beta^P\}_{\beta\in\Gamma_P}$ is a local system of A, where A_β^P is the ideal of A_β that generated by the algebra P for all $\beta \in \Gamma_P$.

Proof. 1. This is proved in [13]. For the proof see [13, Proposition 4.5].

2. By Lemma 3.3, $\{A_\beta\}_{\beta\in\Gamma_P}$ is a local system of A. Let A_β^P be the ideal of A_β that generated by P. We need to show that $\{A_\beta^P\}_{\beta\in\Gamma_P}$ is a local system of A. Since $P \subseteq \cap_{\beta\in\Gamma_P} A_\beta^P$, we have $\cap_{\beta\in\Gamma_P} A_\beta^P \neq 0$, so by (1), for each $\beta \in \Gamma_P$, there is $\gamma \in \Gamma_P$ such that $A_\beta \subseteq A_\gamma^P$. Thus, $L = \cup_{\beta\in\Gamma_P} A_\beta^P$. It remain to note that $A_\beta^P, A_\gamma^P \subseteq A_\zeta^P$. Indeed, for each $\beta, \gamma \in \Gamma_P$, there is $\zeta \in \Gamma_P$ such that $A_\beta, A_\gamma \subseteq A_\zeta$ because $\{A_\beta\}_{\beta\in\Gamma_P}$ is a local system of A. Therefore, $\{A_\beta^P\}_{\beta\in\Gamma_P}$ is a local system of A, as required. □

Proposition 3.5. *Let A be a simple locally finite associative algebra and let $\{A_\alpha\}_{\alpha\in\Gamma}$ be a local system of A. Let S be a non-zero finite dimensional subspace of A and let A_α^s be the ideal of A_α generated by S. Then $\{A_\beta^s\}_{\beta\in\Gamma_s}$ is a local system of A, where $\Gamma_s = \{\beta \in \Gamma \mid S \subseteq A_\beta\}$.*

Proof. This follows directly from Proposition 3.4. □

Recall that an associative algebra A is nilpotent if there is a positive integer n such that $A^n = 0$. Put $A = A^1$ and $A^i = AA^{i-1}$ for all $i > 1$.

Definition 3.6. *1. We say that an associative algebra P over a field $\mathbb{F}$ is* residually nilpotent *if* $\cap_{i=1}^\infty P^i = 0$.

2. We say that a locally finite associative algebra A is locally residually nilpotent *if every finitely generated subalgebra P of A is a residually nilpotent associative algebra.*

Theorem 3.7. *Let A be a simple locally finite associative algebra over $\mathbb{F}$. Then every local system $\{A_\alpha\}_{\alpha\in\Gamma}$ contains an algebra that is not nilpotent.*

Proof. Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be a local system of A. Consider the algebra $P = \mathbb{F}p$, where $p \in A$ is non-zero. Put $\Gamma_p = \{\alpha \in \Gamma \mid A_\alpha \supset P\}$. By Lemma 3.3, $\{A_\alpha\}_{\alpha\in\Gamma_p}$ is a local system of A. Assume to the contrary that all A_α are residually nilpotent for all $\alpha \in \Gamma$. Then for each $\alpha \in \Gamma_p$, there is a positive integer n such that $p \in A_\beta^n$ and $p \notin A_\beta^{n+1}$, so we get system of ideals $\{A_\alpha^n\}_{\alpha\in\Gamma_p}$ with $\cap_{\alpha\in\Gamma_P} A_\alpha^n \neq 0$. By Proposition 3.4, for each $\alpha \in \Gamma_p$, there is $\beta \in \Gamma_p$ such that $A_\alpha \subseteq A_\beta^n$. Thus, $A_\alpha^2 \subseteq A_\beta^{n+1}$, so $p \notin A_\alpha^2$ for all $\alpha \in \Gamma$. Since $p \in A^2$ (because $A = A^2$ as A is simple), there exists $\gamma \in \Gamma$ such that $p = \sum_{i=1}^n x_i y_i$ for some $x_i, y_i \in A_\gamma$ $(1 \le i \le n)$. Thus, $p \in A_\gamma^2$, a contradiction with $p \notin A_\alpha^2$ for each $\alpha \in \Gamma$. Therefore, every local system of A must contain an algebra which is not residually nilpotent, as required. □

Corollary 3.8. *No simple locally finite associative algebra can be locally residually nilpotent.*

Proof. This follows from Theorem 3.7. □

4. Perfect local systems

Recall that a local system is said to be *perfect* (resp. *simple, semisimple, nilpotent, ... etc*) if it consists of perfect (simple, semisimple, nilpotent, ... etc) algebras.

Theorem 4.1. *Any simple locally finite associative algebra posses a perfect local system.*

Proof. Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be a local system of A. Put

$$A_\alpha^\infty = \cap_{i=1}^\infty A_\alpha^i.$$

Then A_α^∞ is a perfect subalgebra of A^∞. Moreover, $\{A_\alpha^\infty\}_{\alpha\in\Gamma}$ is a local system of the subalgebra $A^\infty = \cup_{\alpha\in\Gamma} A_\alpha^\infty$ of A. Indeed, if $\alpha, \beta \in \Gamma$, then there is $\gamma \in \Gamma$ such that $A_\alpha, A_\beta \subseteq A_\gamma$, so $A_\alpha^\infty, A_\beta^\infty \subseteq A_\gamma^\infty$. Note that A^∞ is an ideal of A and $A^\infty \neq 0$ (by Theorem 3.7), but A is simple, so $A^\infty = A$. Therefore, A contains a perfect local system, as required. □

Lemma 4.2. *Let A be a simple locally finite associative algebra over $\mathbb{F}$. The following holds:*

1. *[3] Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be a local system of A. Then for every $\alpha \in \Gamma$, there is $\zeta \in \Gamma$ such that $A_\alpha \subset A_\zeta$ and $A_\alpha \bigcap \operatorname{rad} A_\zeta = 0$.*

2. *[9] If $\{A_\alpha\}_{\alpha\in\Gamma}$ is a perfect local system of A, then there exists $\alpha' \in \Gamma$ for each $\alpha \in \Gamma$ such that $\operatorname{rad} A_\beta \bigcap A_\alpha = 0$ for all $\beta \geq \alpha'$.*

Proof. 1. Let $A_\alpha \in \{A_\alpha\}_{\alpha\in\Gamma}$. Suppose that $\Gamma_\alpha = \{\beta \in \Gamma \mid A_\alpha \subseteq A_\beta\}$. By Lemma 3.3, $\{A_\beta\}_{\beta\in\Gamma_P}$ is a local system of A. Let $\{\operatorname{rad} A_\beta\}_{\beta\in\Gamma_\alpha}$ be the system of the radicals of $\{A_\beta\}_{\beta\in\Gamma_\alpha}$ such that $\operatorname{rad} A_\beta$ is the radical of A_β. By Theorem 3.7, not all members of $\{A_\beta\}_{\beta\in\Gamma_P}$ are nilpotent algebras, so there exists $\gamma \in \Gamma_\alpha$ such that $A_\gamma \nsubseteq \operatorname{rad} A_\beta$ for all $\beta \in \Gamma_\alpha$. Thus, $\cap_{\beta\in\Gamma_\alpha} \operatorname{rad} A_\beta = 0$ (by Proposition 3.4).

Now, we have $A_\alpha \cap \operatorname{rad} A_\beta \subseteq \operatorname{rad} A_\alpha$ for all $\beta \in \Gamma_\alpha$. Since $\dim(\operatorname{rad} A_\beta)$ is finite, there exist $\beta_1, \beta_2, \ldots, \beta_n$ $(n \geq 1)$ such that $A_\alpha \cap (\cap_{i=1}^n \operatorname{rad} A_{\beta_i}) = 0$. Now, take any $\zeta \in \Gamma_\alpha$ such that $A_\zeta \supseteq A_{\beta_1}, \ldots, A_{\beta_n}$. Then $A_{\beta_i} \cap \operatorname{rad} A_\zeta \subseteq \operatorname{rad} A_{\beta_i}$ for all $1 \leq i \leq n$, so

$$A_\alpha \cap \operatorname{rad} A_\zeta \subseteq A_{\beta_i} \cap \operatorname{rad} A_\zeta \subseteq \operatorname{rad} A_{\beta_i}, \quad \text{for all } 1 \leq i \leq n.$$

Therefore,

$$A_\alpha \cap \operatorname{rad} A_\zeta \subseteq A_\alpha \cap (\cap_{i=1}^n \operatorname{rad} A_{\beta_i}) = 0,$$

as required.

2. This follows from part (1.). □

Theorem 4.3. *Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be a local system of a simple locally finite associative algebra A over $\mathbb{F}$. Then for each $\alpha \in \Gamma$, there is $\gamma \in \Gamma$ such that $A_\alpha \subset A_\gamma$ and $M_\gamma \cap A_\alpha = 0$ for some maximal ideal M_γ of A_γ.*

Proof. By Lemma 4.2, A has a perfect local system. Suppose that $\{A_\alpha\}_{\alpha\in\Gamma}$ is a perfect local system of A. Fix any $\alpha \in \Gamma$. By Lemma 4.2(2), there is $\zeta \in \Gamma$ such that

$$A_\alpha \subset A_\zeta \quad \text{and} \quad A_\alpha \cap \operatorname{rad} A_\zeta = 0. \tag{3}$$

Let $\Gamma^\zeta = \{\beta \in \Gamma \mid \beta \geq \zeta\}$. Then by Lemma 3.2,

$$\{A_\beta\}_{\beta\in\Gamma^\zeta} = \{A_\beta \in \{A_\alpha\}_{\alpha\in\Gamma} \mid A_\zeta \subset A_\beta\}$$

is a local system of A. Let $\{P_{\zeta_1}, \ldots, P_{\zeta_t}\}$ and $\{M_{\zeta_1}, \ldots, M_{\zeta_r}\}$ be the set of all minimal perfect and maximal ideals of A_ζ, respectively. Consider two subsets $\Gamma^\zeta_{l,k}$ and Γ^ζ_0 of Γ^ζ defined as follows:

$$\Gamma^\zeta_{k,l} = \{\beta \in \Gamma^\zeta \mid A_\beta \quad \text{has an ideal} \quad I_\beta \quad \text{with} \quad P_{\zeta_k} \subset I_\beta \cap A_\zeta \subset M_{\zeta_l}, 1 \leq k \leq t, 1 \leq l \leq r\};$$

$$\Gamma^\zeta_0 = \{\beta \in \Gamma^\zeta \mid \text{any proper ideal} \quad I_\beta \quad \text{of} \quad A_\beta \quad \text{satisfies} \quad I_\beta \cap A_\zeta \subset \operatorname{rad} A_\zeta.$$

Then $\Gamma^\zeta = \Gamma^\zeta_{k,l} \cup \Gamma^\zeta_0$, so by Lemma 3.2, either $\{A_\beta\}_{\beta\in\Gamma^\zeta_{k,\ell}}$ or $\{A_\beta\}_{\beta\in\Gamma^\zeta_0}$ is a local system of A. We claim that $\{A_\beta\}_{\beta\in\Gamma^\zeta_0}$ is a local system of A. Assume to the contrary that $\{A_\beta\}_{\beta\in\Gamma^\zeta_{k,l}}$ is local system. Then for any $A_\gamma \in \{A_\beta\}_{\beta\in\Gamma^\zeta_{k,l}}$, there is an ideal $I_\gamma \subset A_\gamma$ such that $P_{\zeta_k} \subset I_\gamma \cap A_\zeta \subseteq M_{\zeta_l}$ for some k and l. Hence, we get a system of ideals $\{I_\beta\}_{\beta\in\Gamma^\zeta_{k,\ell}}$ such that I_β is an ideal of A_β with $P_{\zeta_k} \subset I_\beta \cap A_\zeta \subset M_{\zeta_l}$ for some $1 \leq k \leq t$ and $1 \leq l \leq r$. Note that I_β $\nsupseteq A_\zeta$ for all $\beta \in \Gamma^\zeta_{k,l}$ (because if $A_\zeta \subseteq I_\beta$, then $A_\zeta = I_\beta \cap A_\zeta \subset M_{\zeta_l}$, a contradiction as M_{ζ_l} is a proper maximal ideal of A_ζ. Since $A_\zeta \subseteq A_\beta$, for each $\beta \in \Gamma^\zeta_{k,l}$, there is no $\delta_\beta \in \Gamma^\zeta_{k,l}$ such that $A_\beta \subseteq I_{\delta_\beta}$, so by Proposition 3.4,

$$\cap_{\beta\in\Gamma^\zeta_{k,l}} I_\beta = 0, \quad \text{but } \cap_{\beta\in\Gamma^\zeta_{k,l}} I_\beta \in \{I_\beta\}_{\beta\in\Gamma^\zeta_{k,l}},$$

so there is some $1 \leq k \leq t$ such that $0 \neq P_{\zeta_k} \subset \cap_{\beta \in \Gamma_{k,l}} I_\beta$, a contradiction.

Hence, $\{A_\beta\}_{\beta \in \Gamma_0^\zeta}$ is a local system of A. Thus, for every proper ideal I_γ of $A_\gamma \in \{A_\beta\}_{\beta \in \Gamma_0^\zeta}$, we have $I_\gamma \cap A_\zeta \subseteq \operatorname{rad} A_\zeta$, so for $\alpha \in \Gamma$, there is $\gamma \in \Gamma_0^\zeta \subset \Gamma^\zeta \subset \Gamma$ such that if I_γ is a proper ideal of A_γ, then

$$I_\gamma \cap A_\alpha = (M_\gamma \cap A_\zeta) \cap A_\alpha \subset \operatorname{rad} A_\zeta \cap A_\alpha = 0.$$

Therefore, for each $\alpha \in \Gamma$, there exists $\gamma \in \Gamma$ such that $A_\alpha \subset A_\gamma$ and $M_\gamma \cap A_\alpha = 0$ for some maximal ideal M_γ of A_γ, as required. □

If A is finite dimensional, then it follows by Wedderburn-Malcev Theorem (see [8, Theorem 1]) that there exists a semisimple subalgebra S of A such that $A = S \oplus \operatorname{rad} A$ and for any semisimple subalgebra Q of A, there is $r \in \operatorname{rad} A$ with $Q \subseteq (1+r)S(1+r)$.

Theorem 4.4. *[8, Theorem 6] Let A be a finite dimensional algebra and let I be a left ideal of A. Suppose that $A/RadA$ is separable. Then there exists a semisimple subalgebra S of A such that $A = S \oplus \operatorname{rad} A$ and $I = I_S \oplus I_{\operatorname{rad} A}$, where $I_S = I \bigcap S$ and $I_{\operatorname{rad} A} = I \bigcap \operatorname{rad} A$.*

Recall that the *rank of a perfect finite dimensional algebra* A is the smallest rank of the simple components of $A/\operatorname{rad} A$.

Theorem 4.5. *Every simple locally finite associative algebra over $\mathbb{F}$ has a perfect local system of arbitrary large rank.*

Proof. Let A be a simple locally finite associative algebra. Then by Theorem 4.1, A has a perfect local system. Let $\{A_\alpha\}_{\alpha \in \Gamma}$ be a perfect local system of A. Then by Theorem 4.3, for each $\alpha \in \Gamma$, there is $\gamma \in \Gamma$ such that $A_\alpha \subset A_\gamma$ and $M_\gamma \cap A_\alpha = 0$ for some maximal ideal M_γ of A_γ. Since A_γ is finite dimensional, $A_\gamma = S_\gamma \oplus R_\gamma$, where S_γ is a Levi subalgebra of A_γ and $R_\gamma = \operatorname{rad} A_\gamma$ is the radical of A_γ. Let $\{S_\gamma^1, \ldots, S_\gamma^n\}$ be the set of the simple components of S_γ. First we claim that $R_\gamma \subseteq M_\gamma$. Assume to the contrary that $R_\gamma \nsubseteq M_\gamma$. Then $M_\gamma + R_\gamma \neq M_\gamma$. Since M_γ is maximal, $A_\gamma = M_\gamma + R_\gamma$. Thus,

$$A_\gamma/M_\gamma = (M_\gamma + R_\gamma)/M_\gamma \cong R_\gamma/(R_\gamma \cap M_\gamma) \neq 0$$

is a non-zero nilpotent quotient algebra of A_γ, but $A_\gamma^2 = A_\gamma$ (as A_γ are perfect for all γ), so $A_\gamma = A_\gamma^n \subseteq M_\gamma$, a contradiction. Thus, $R_\gamma \subset M_\gamma$. Note that $M_\gamma \not\supseteq S_\gamma$ (because $M_\gamma \neq A_\gamma$). Since

$$A_\gamma/M_\gamma = (S_\gamma \oplus R_\gamma)/M_\gamma = (S_\gamma + M_\gamma)/M_\gamma \cong S_\gamma/(S_\gamma \cap M_\gamma) \neq 0,$$

$A_\gamma/M_\gamma \cong S_\gamma^i$ for some $1 \leq i \leq n$. Since $A_\alpha \cap M_\gamma = 0$, we have that $A_\alpha \subseteq S_\gamma^i$. Therefore, there is a simple component S_γ^i in every Levi subalgebra S_γ of A_γ such that $\dim S_\gamma^i \geq \dim A_\gamma$. □

5. 1-perfect and conical local systems

Definition 5.1. *[10] Let A be an associative algebra over a field $\mathbb{F}$. Then A is called* 1-perfect *if A has no ideals of codimension 1.*

An ideal I of A is called 1-perfect *if as an algebra I is 1-perfect. By using the second and the third Isomorphism Theorems, we obtain the following well known properties.*

Lemma 5.2. *[10] (i) The sum of 1-perfect ideals of an associative algebra A is a 1-perfect ideal of A.*

(ii) Let P be a 1-perfect ideal of A and let C be a 1-perfect ideal of A/P. Then the full preimage of C in A is 1-perfect.

By using Lemma 5.2(i) we get that every associative algebra contains the largest 1-perfect ideal.

Definition 5.3. *Let A be an associative algebra and let $\mathcal{P}_A$ be the largest 1-perfect ideal of A. Then $\mathcal{P}_A$ is said to be* the 1-perfect radical *of A.*

We will need the following simple fact.

Lemma 5.4. *Let P be an ideal of an associative algebra A over a field $\mathbb{F}$. If P' is an ideal of P with $P'^2 = P'$, then P' is an ideal of A.*

The following results due to Baranov and Shlaka [10] Shows that $\mathcal{P}_A$ has radical-like properties.

Proposition 5.5. *Let A be an associative algebra over $\mathbb{F}$. The following hold.*

(1) $\mathcal{P}_A^2 = \mathcal{P}_A$.

(2) $\mathcal{P}_{\mathcal{P}_A} = \mathcal{P}_A$.

(3) $\mathcal{P}_{A/\mathcal{P}_A} = 0$.

(4) Consider any maximal chain of subalgebras $A = A_0 \supset A_1 \supset \cdots \supset A_r$ of A such that A_{i+1} is an ideal of A_i. If $\dim A_i/A_{i+1} = 1$ for all $0 \le i \le r-1$, then $A_r = \mathcal{P}_A$.

Theorem 5.6. *Any simple locally finite associative algebra posses a 1-perfect local system.*

Proof. By Theorem 4.1 A contains a perfect system. Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be a perfect system of A. Consider the largest 1-perfect radical $\mathcal{P}_{A_\alpha}$ of A_α for each $\alpha \in \Gamma$. Since $\mathcal{P}_{A_\alpha}$ is finite dimensional, $\mathcal{P}_{A_\alpha} = S_\alpha \oplus \operatorname{rad}\mathcal{P}_{A_\alpha}$ for some Levi subalgebra S_α of $\mathcal{P}_{A_\alpha}$. Let $\{S_\alpha^1, \dots, S_\alpha^n\}$ be the set of the simple components of S_α. We denote by A_α^i to be the ideal of $\mathcal{P}_{A_\alpha}$ generated by S_α^i. Fix any index, say 1. Since A_α^1 is perfect, by Lemma 5.4, A_α^1 is an ideal of A_α. Thus, by Proposition 3.5, $\{A_\alpha^1\}_{\alpha\in\Gamma_{S_\alpha^1}}$ is a 1-perfect local system of A, where $\Gamma_{S_\alpha^1} = \{\beta \in \Gamma \mid A_\beta \supset A_\alpha^1\}$. □

Definition 5.7. *Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be a perfect local system of A. Then $\{A_\alpha\}_{\alpha\in\Gamma}$ is said to be* conical *if Γ contains a minimal element 1 such that*

1. *$A_1 \subseteq A_\alpha$ for all $\alpha \in \Gamma$;*
2. *A_1 is simple;*
3. *for each $\alpha \in \Gamma$ the restriction of any A_α-module to A_1 has a non-zero composition factor.*

Note that Definition 5.7(3), implies that for each simple component T of $A_\alpha/\operatorname{rad} A_\alpha$ one has $\dim T \ge \dim A_\alpha$. We denote by the *rank of the conical local system* is the rank of the perfect algebra A_1.

Theorem 5.8. *Every simple locally finite associative algebra over $\mathbb{F}$ has a conical local system of arbitrary large rank.*

Proof. Let A be a simple locally finite associative algebra over $\mathbb{F}$. By Corollary 4.5, A has a perfect local system of arbitrary large rank. Suppose that $\{A_\alpha\}_{\alpha\in\Gamma}$ is a perfect local system of A of arbitrary large rank. Fix $A_\beta \in \{A_\alpha\}_{\alpha\in\Gamma}$. Since A_β is finite dimensional associative algebra, there is a Levi subalgebra S_β of A_β such that $A_\beta = S_\beta \oplus R_\beta$, where $R_\beta = \operatorname{rad} A_\beta$ is the radical of A_β. Let S be a simple component of S_β. For all $\gamma \ge \beta$ we denote by A_γ^s to be the ideal of A_γ that generated by S. Then A_γ^s is the smallest ideal of A_γ that contains S. Since $(A_\gamma^s)^2 \subseteq A_\gamma^s$ is also an ideal of A_γ that contains S, we get that $(A_\gamma^s)^2 = A_\gamma^s$, so A_γ^s is perfect. Put $A_1^s = S$ and $\gamma^s = \{\gamma \in \gamma \mid \gamma \ge \beta\} \cup \{1\}$. Put $A^s = \cup_{\gamma\in\gamma^s} A_\gamma^s$. Since $A_\gamma^s \subseteq A_{\gamma'}^s$ for all γ and $\gamma' \in \gamma^s$ with $\gamma \le \gamma'$, we get that $A^s = \lim_{\longleftarrow} A_\gamma^s$ is an ideal of A. But A is simple, so $A^s = A$ and $\{A_\gamma^s\}_{\gamma\in\gamma^s}$ is a perfect local system of A. Moreover, we have $\{A_\gamma^s\}_{\gamma\in\gamma^s}$ is a conical local system of A because $\{A_\gamma^s\}_{\gamma\in\gamma^s}$ is a perfect local system with γ^s containing a minimal element 1 that satisfies the conditions (1), (2) and (3) of Definition 5.7, so it is a local system of arbitrary large rank, as required. □

Theorem 5.9. *Every simple locally finite associative algebra over $\mathbb{F}$ has 1-perfect conical local system .*

Proof. Let A be a simple locally finite associative algebra over $\mathbb{F}$. By Theorem 5.6, A has a 1-perfect local system. It remains to follow the same process as in the proof of Theorem 5.8, we get the required results. $\square$

Disclosure statement

Data Availability Statement: The authors declare that [the/all other] data supporting the findings of this study are available within the article. Any clarification may be requested from the corresponding author, provided it is essential.
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