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On the three graph invariants related to matching of finite simple graphs*

Research Article

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Abstract: Let G be a finite simple graph on the vertex set $V(G)$ and let ind-match(G), min-match(G) and match(G) denote the induced matching number, the minimum matching number and the matching number of G, respectively. It is known that the inequalities ind-match$(G) \leq$ min-match$(G) \leq$ match$(G) \leq$ 2min-match(G) and match$(G) \leq \lfloor |V(G)|/2 \rfloor$ hold in general. In the present paper, we determine the possible tuples (p, q, r, n) with ind-match$(G) = p$, min-match$(G) = q$, match$(G) = r$ and $|V(G)| = n$ arising from connected simple graphs. As an application of this result, we also determine the possible tuples (p', q, r, n) with reg$(G) = p'$, min-match$(G) = q$, match$(G) = r$ and $|V(G)| = n$ arising from connected simple graphs, where $I(G)$ is the edge ideal of G and reg$(G) = $ reg$(K[V(G)]/I(G))$ is the Castelnuovo–Mumford regularity of the quotient ring $K[V(G)]/I(G)$.

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1. Introduction

Let $G = (V(G), E(G))$ be a finite simple graph on the vertex set $V(G)$ with the edge set $E(G)$. The main topic of this paper is graph-theoretical invariants related to *matching*.

- A subset $M = \{e_1, \ldots, e_s\} \subset E(G)$ is said to be a *matching* of G if $e_i \cap e_j = \emptyset$ for all $1 \leq i < j \leq s$. For a matching M, we write $V(M) = \{v : v \in e \text{ for some } e \in M\}$. A *perfect matching* M is a matching of G with $V(M) = V(G)$.

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- A matching of M of G is *maximal* if $M \cup \{e\}$ cannot be a matching of G for all $e \in E(G) \setminus M$. Note that a matching M is maximal if and only if $V(G) \setminus V(M)$ is an independent set of G.
- A matching $M = \{e_1, \ldots, e_s\} \subset E(G)$ of G is said to be an *induced matching* if, for all $1 \leq i < j \leq s$, there is no edge $e \in E(G)$ with $e \cap e_i \neq \emptyset$ and $e \cap e_j \neq \emptyset$.
- The *matching number* match(G), the *minimum matching number* min-match(G) and the *induced matching number* ind-match(G) of G are defined as follows respectively:

$$\begin{aligned} \text{match}(G) &= \max\{|M| : M \text{ is a matching of } G\}; \\ \text{min-match}(G) &= \min\{|M| : M \text{ is a maximal matching of } G\}; \\ \text{ind-match}(G) &= \max\{|M| : M \text{ is an induced matching of } G\}. \end{aligned}$$

Hall's marriage theorem [16] says that, for any bipartite graph G on the bipartition $V(G) = X \cup Y$ with $|X| \leq |Y|$, $\text{match}(G) = |X|$ holds if and only if $|N_G(S)| \geq |S|$ for all $S \subset X$, where $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G(v) = \{w \in V(G) : \{v, w\} \in E(G)\}$. There are many previous studies for $\text{match}(G)$, $\text{min-match}(G)$ and $\text{ind-match}(G)$, see [1–3, 9, 12, 31, 34]. To explain our motivation, we focus on two known results described below.

First, in [18], it is proven that the inequalities

$$\text{ind-match}(G) \leq \text{min-match}(G) \leq \text{match}(G) \leq 2\text{min-match}(G) \tag{1}$$

hold for all simple graph G and a classification of connected simple graphs G satisfying $\text{ind-match}(G) = \text{min-match}(G) = \text{match}(G)$ is given ([4, Theorem 1], [17, Remark 0.1]) and such graphs are studied in [17, 19, 21, 22, 32, 36]. A classification of connected simple graphs G with $\text{ind-match}(G) = \text{min-match}(G)$ is also given in [18].

Second, by definition of a matching, the equality

$$\text{match}(G) \leq \lfloor |V(G)|/2 \rfloor \tag{2}$$

holds and some classes of graphs G with $\text{match}(G) = \lfloor |V(G)|/2 \rfloor$ are given (see [5, 13, 27, 35]).

From (1) and (2), it is natural to ask the following question:

Question A: Is there any connected simple graph $G = G(p, q, r, n)$ such that

$$\text{ind-match}(G) = p, \text{min-match} = q, \text{match}(G) = r \textit{ and } |V(G)| = n$$

for all integers p, q, r, n with $1 \leq p \leq q \leq r \leq 2q$ and $r \leq \lfloor n/2 \rfloor$?

As a previous study related to Question A, a connected simple graph $G = G(p, q, r)$ such that $\text{ind-match}(G) = p, \text{min-match}(G) = q$ and $\text{match}(G) = r$ was constructed for all integers p, q, r with $1 \leq p \leq q \leq r \leq 2q$ (see [18, Theorem 2.3]). Since the graph G constructed in the proof of [18, Theorem 2.3] satisfies $|V(G)| = 2\text{match}(G) + \text{ind-match}(G) - 1$, for example, we can see that there exists a connected simple graph $G = G(2, 3, 4, 9)$ such that $\text{ind-match}(G) = 2, \text{min-match}(G) = 3, \text{match}(G) = 4$ and $|V(G)| = 9$ by virtue of [18, Theorem 2.3]. However, there is a connected simple graph $G = G(2, 3, 4, 8)$ such that $\text{ind-match}(G) = 2, \text{min-match}(G) = 3, \text{match}(G) = 4$ and $|V(G)| = 8$; see Figure 1. This graph says that it is necessary to construct new families of connected simple graphs in order to solve Question A.

Based on the above, we state main results of the present paper. As the first result is, we determine the possible tuples

$$(\text{ind-match}(G), \text{min-match}(G), \text{match}(G), |V(G)|)$$

arising from connected simple graphs.

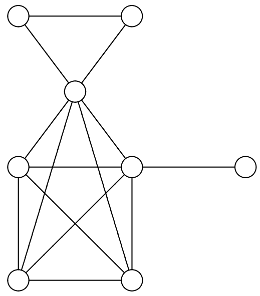

Figure 1. A connected graph $G = G(2, 3, 4, 8)$ with ind-match$(G) = 2$, min-match$(G) = 3$, match$(G) = 4$ and $|V(G)| = 8$.

Theorem 1.1. *(see Theorem 3.1) Let $n \geq 2$ be an integer and set*

$$\begin{aligned}&\mathbf{Graph}_{\text{ind-match,min-match,match}}(n)\\ &= \left\{(p,q,r) \in \mathbb{N}^3 \;\middle|\; \begin{array}{l}\text{There exists a connected simple graph } G \text{ with } |V(G)| = n\\ \text{and ind-match}(G) = p,\ \text{min-match}(G) = q,\ \text{match}(G) = r\end{array}\right\}.\end{aligned}$$

Then we have the following:

(1) *If n is odd, then*

$$\begin{aligned}&\mathbf{Graph}_{\text{ind-match,min-match,match}}(n)\\ &= \left\{(p,q,r) \in \mathbb{N}^3 \;\middle|\; 1 \leq p \leq q \leq r \leq 2q \ \text{ and } \ r \leq \frac{n-1}{2}\right\}.\end{aligned}$$

(2) *If n is even, then*

$$\begin{aligned}&\mathbf{Graph}_{\text{ind-match,min-match,match}}(n)\\ &= \left\{(1,q,r) \in \mathbb{N}^3 \;\middle|\; 1 \leq q \leq r \leq 2q \ \text{ and } \ r \leq \frac{n}{2}\right\}\\ &\cup \left\{(p,q,r) \in \mathbb{N}^3 \;\middle|\; 2 \leq p \leq q \leq r \leq 2q,\ r \leq \frac{n}{2} \ \text{ and } \ (q,r) \neq \left(\frac{n}{2}, \frac{n}{2}\right)\right\}.\end{aligned}$$

Note that this theorem gives a complete answer for Question A.

The second main result is related to the invariants of edge ideals. Let G be a finite simple graph on the vertex set $V(G) = \{x_1, \ldots, x_{|V(G)|}\}$ and $E(G)$ the set of edges of G. Let $K[V(G)] = K\left[x_1, \ldots, x_{|V(G)|}\right]$ be the polynomial ring in $|V(G)|$ variables over a field K. The *edge ideal* of G, denoted by $I(G)$, is the ideal of $K[V(G)]$ generated by quadratic monomials $x_i x_j$ associated with $\{x_i, x_j\} \in E(G)$. Among the current trends in combinatorial commutative algebra, the edge ideal is one of the main topic and has been studied extensively by many researchers, see [6–8, 15, 26, 28, 29, 33, 37, 38].

Let $\mathrm{reg}(G) = \mathrm{reg}(K[V(G)]/I(G))$ denote the *Castelnuovo–Mumford regularity* (regularity for short, see [30, Section 18]) of the quotient ring $K[V(G)]/I(G)$. It is known that

$$\text{ind-match}(G) \leq \mathrm{reg}(G) \leq \text{min-match}(G)$$

holds for all simple graph G (the lower bound was given by Katzman [25] and the upper bound was given by Woodroofe [38]). Moreover, Hà–Van Tuyl proved that ind-match$(G) = \mathrm{reg}(G)$ holds if G is a chordal graph ([15, Corollary 6.9]). By virtue of this result together with Theorem 1.1, we also determine the possible tuples

$$(\mathrm{reg}(G), \text{min-match}(G), \text{match}(G), |V(G)|)$$

arising from connected simple graphs. The second main result is as follows.

Theorem 1.2. *(see Theorem 4.1) Let $n \geq 2$ be an integer and set*

$$\begin{aligned} &\mathbf{Graph}_{\text{reg,min-match,match}}(n) \\ &= \left\{ (p', q, r) \in \mathbb{N}^3 \;\middle|\; \begin{array}{c} \text{There exists a connected simple graph } G \text{ with } |V(G)| = n \\ \text{and } \ \operatorname{reg}(G) = p', \ \text{min-match}(G) = q, \ \text{match}(G) = r \end{array} \right\}. \end{aligned}$$

Then one has

$$\mathbf{Graph}_{\text{reg,min-match,match}}(n) = \mathbf{Graph}_{\text{ind-match,min-match,match}}(n).$$

Our paper is organized as follows. In Section 2, we prepare some lemmas and propositions in order to prove main results. In Section 3, we give a proof of Theorem 1.1. In Section 4, we introduce previous studies related to Theorem 1.2 and give a proof.

2. Preparation

In this section, we prepare some lemmas and propositions in order to prove our main results.

2.1. Known results

In this subsection, we present known results related to our study. First, we recall two important inequalities.

Proposition 2.1. *Let G be a finite simple graph on the vertex set $V(G)$. Then*

(1) $\text{ind-match}(G) \leq \text{min-match}(G) \leq \text{match}(G) \leq 2\text{min-match}(G)$.

(2) $\text{match}(G) \leq \lfloor |V(G)|/2 \rfloor$.

Proof. (1) : See [18, Proposition 2.1 and Remark 3.2].
(2) : Let M be a matching of G with $|M| = \text{match}(G)$. Then $|V(G)| \geq |V(M)| = 2\text{match}(G)$ by the definition of matching. Hence we have $\text{match}(G) \leq \lfloor |V(G)|/2 \rfloor$. □

Next, we recall the definition of the S-suspension introduced in [20]. A subset $S \subset V(G)$ is said to be an *independent set* of G if $\{u, v\} \notin E(G)$ for all $u, v \in S$. Note that the empty set $\emptyset$ is an independent set of G. For an independent set S of G, we define the graph G^S as follows:

- $V(G^S) = V(G) \cup \{w\}$, where w is a new vertex.
- $E(G^S) = E(G) \cup \{\{v, w\} : v \notin S\}$.

We call G^S the *S-suspension* of G.

Lemma 2.2 ([20, Lemma 1.5]). *Let G be a finite simple graph on the vertex set $V(G)$. Suppose that G has no isolated vertices. Let $S \subset V(G)$ be an independent set of G. Then* $\text{ind-match}(G^S) = \text{ind-match}(G)$ *holds.*

To end this subsection, we recall a classification of connected simple graphs G with $\text{min-match}(G) = |V(G)|/2$ given by Arumugam–Velammal.

Proposition 2.3 ([2, Theorem 2.1]). *Let $n \geq 2$ be an even integer and let $G = (V(G), E(G))$ be a connected simple graph with $|V(G)| = n$. Assume that* $\text{min-match}(G) = n/2$. *Then G is either K_n or $K_{n/2,n/2}$. In particular,* $\text{ind-match}(G) = 1$ *and there is no connected simple graph G with* $(\text{ind-match}(G), \text{min-match}(G), \text{match}(G)) = (p, n/2, n/2)$ *and $|V(G)| = n$ for all $p \geq 2$.*

2.2. Induced subgraph

Let W be a subset of $V(G)$. We define the *induced subgraph* G_W of G on W as follows:

- $V(G_W) = W$.
- $E(G_W) = \{\{u, v\} \in E(G) \mid u, v \in W\}$.

For a vertex $v \in V(G)$, we denote $G_{V(G)\setminus\{v\}}$ by $G \setminus v$.

Proposition 2.4. *Let G be a finite simple graph on the vertex set $V(G)$ and $v \in V(G)$. Then*

(1) $\text{ind-match}(G \setminus v) \leq \text{ind-match}(G)$.

(2) $\text{min-match}(G \setminus v) \leq \text{min-match}(G)$.

(3) $\text{match}(G \setminus v) \leq \text{match}(G)$.

Proof. (1), (3) : it is easy to prove since induced matchings (resp. matchings) of $G \setminus \{v\}$ are induced matching (resp. matching) of G.
(2) : Let $M \subset E(G)$ be a maximal matching with $|M| = \text{min-match}(G)$. Note that $V(G) \setminus V(M)$ is an independent set of G.

- Assume that $v \in V(G) \setminus V(M)$. Then $\{V(G) \setminus V(M)\} \setminus \{v\}$ is an independent set of $G \setminus v$. Hence M is a maximal matching of $G \setminus v$ since $V(G \setminus v) \setminus V(M) = \{V(G) \setminus V(M)\} \setminus \{v\}$. Thus, we have $\text{min-match}(G \setminus v) \leq |M| = \text{min-match}(G)$.
- Assume that $v \notin V(G) \setminus V(M)$. Then $v \in V(M)$ and there exists an edge $\{v, w\} \in M$.
  - Assume that there exists a vertex $w' \in V(G) \setminus V(M)$ such that $\{w, w'\} \in E(G)$. Then $\{V(G) \setminus V(M)\} \setminus \{w'\}$ is an independent set of $G \setminus v$. Now we put $M_1 = (M \setminus \{\{v, w\}\}) \cup \{\{w, w'\}\}$. Since $V(G \setminus v) \setminus V(M_1) = \{V(G) \setminus V(M)\} \setminus \{w'\}$, it follows that M_1 is a maximal matching of $G \setminus v$. Hence we have $\text{min-match}(G \setminus v) \leq |M_1| = |M| = \text{min-match}(G)$.
  - Assume that $\{w, w''\} \notin E(G)$ for all $w'' \in V(G) \setminus V(M)$. Then $\{V(G) \setminus V(M)\} \cup \{w\}$ is an independent set of $G \setminus v$. Put $M_2 = M \setminus \{\{v, w\}\}$. Then M_2 is a maximal matching of $G \setminus v$ since $V(G \setminus v) \setminus V(M_2) = \{V(G) \setminus V(M)\} \cup \{w\}$. Hence one has $\text{min-match}(G \setminus v) \leq |M_2| = |M| - 1 < \text{min-match}(G)$.

Therefore we have the desired conclusion. □

As a corollary of Proposition 2.4, one has

Corollary 2.5. *Let G be a finite simple graph on the vertex set $V(G)$ and let $W \subset V(G)$ be a subset. Then one has*

(1) $\text{ind-match}(G_W) \leq \text{ind-match}(G)$.

(2) $\text{min-match}(G_W) \leq \text{min-match}(G)$.

(3) $\text{match}(G_W) \leq \text{match}(G)$.

Lemma 2.6. *Let G be a finite disconnected simple graph and $G_1, \ldots, G_s$ $(s \geq 2)$ the connected components of G. Then we have*

(1) $\text{ind-match}(G) = \displaystyle\sum_{i=1}^{s} \text{ind-match}(G_i)$.

(2) $\text{min-match}(G) = \sum_{i=1}^{s} \text{min-match}(G_i)$.

(3) $\text{match}(G) = \sum_{i=1}^{s} \text{match}(G_i)$.

Proof. It is enough to show the case $s = 2$.
(1) : For $1 \leq i \leq 2$, let $M_i \subset E(G_i)$ be an induced matching of G_i with $|M_i| = \text{ind-match}(G_i)$. Since $M_1 \cup M_2$ is an induced matching of G and $M_1 \cap M_2 = \emptyset$, one has

$$\text{ind-match}(G_1) + \text{ind-match}(G_2) = |M_1| + |M_2| = |M_1| \cup |M_2| \leq \text{ind-match}(G).$$

Next, we show the opposite inequality. Let M be an induced matching of G with $|M| = \text{ind-match}(G)$. Since each $e \in M$ is an edge of G_1 or G_2 and $E(G_1) \cap E(G_2) = \emptyset$, there exist $M_1 \subset E(G_1)$ and $M_2 \subset E(G_2)$ such that $M_1 \cup M_2 = M$ and $M_1 \cap M_2 = \emptyset$. Note that M_1 (resp. M_2) is an induced matching of G_1 (resp. G_2). Hence it follows that $|M_i| \leq \text{ind-match}(G_i)$ for $i = 1, 2$. Thus one has

$$\text{ind-match}(G) = |M| = |M_1 \cup M_2| = |M_1| + |M_2| \leq \text{ind-match}(G_1) + \text{ind-match}(G_2).$$

Therefore we have the desired conclusion.
(2) : Let $M'_i \subset E(G_i)$ be a maximal matching of G_i with $|M'_i| = \text{min-match}(G_i)$ for $i = 1, 2$. Then $M'_1 \cap M'_2 = \emptyset$ and $V(G_i) \setminus V(M'_i)$ is an independent set of G_i for $i = 1, 2$. Since G_1 and G_2 are connected components of G, it follows that $\{V(G_1) \setminus V(M'_1)\} \cup \{V(G_2) \setminus V(M'_2)\}$ is an independent set of G. Hence $M'_1 \cup M'_2$ is a maximal matching of G since $V(G) \setminus V(M'_1 \cup M'_2) = \{V(G_1) \setminus V(M'_1)\} \cup \{V(G_2) \setminus V(M'_2)\}$. Thus we have

$$\text{min-match}(G_1) + \text{min-match}(G_2) = |M'_1| + |M'_2| = |M'_1 \cup M'_2| \geq \text{min-match}(G).$$

Next, we show the opposite inequality. Let M' be a maximal matching of G with $|M'| = \text{min-match}(G)$. Since each $e \in M'$ is an edge of G_1 or G_2 and $E(G_1) \cap E(G_2) = \emptyset$, there exist $M'_1 \subset E(G_1)$ and $M'_2 \subset E(G_2)$ such that $M' = M'_1 \cup M'_2$ and $M'_1 \cap M'_2 = \emptyset$. Note that M'_i is a matching of G_i for $i = 1, 2$. Assume that M'_1 is not maximal. Then there exists $e' \in E(G_1) \setminus M'_1$ such that $M'_1 \cup \{e'\}$ is a matching of G_1. However this is a contradiction because $M' \cup \{e'\}$ is a matching of G in this situation. Hence we have M'_1 is a maximal matching of G_1. Similarly, we also have M'_2 is a maximal matching of G_2. Thus it follows that

$$\text{min-match}(G_1) + \text{min-match}(G_2) \leq |M'_1| + |M'_2| = |M'_1 \cup M'_2| = \text{min-match}(G).$$

Therefore we have the desired conclusion.
(3) : We can prove this by replacing "an induced matching" with "a matching" and "ind-match" with "match" in the proof of (1). ☐

Lemma 2.7. *Let $G = (V(G), E(G))$ be a finite simple graph. Assume that there exist two edges $\{u, w\}, \{v, w\} \in E(G)$ such that $\deg(u) = \deg(v) = 1$. Then*

(1) $\text{ind-match}(G \setminus v) = \text{ind-match}(G)$.

(2) $\text{min-match}(G \setminus v) = \text{min-match}(G)$.

(3) $\text{match}(G \setminus v) = \text{match}(G)$.

Proof. By virtue of Corollary 2.5, it is enough to show "$\geq$" since $G \setminus v$ is an induced subgraph of G.
(1) : Let M be an induced matching of G with $|M| = \text{ind-match}(G)$.

- Assume that $\{v, w\} \notin M$. Then M is also an induced matching of $G \setminus v$. Hence one has $\text{ind-match}(G \setminus v) \geq |M| = \text{ind-match}(G)$.

- Assume that $\{v, w\} \in M$. Then $(M \setminus \{v, w\}) \setminus \{u, w\}$ is an induced matching of $G \setminus v$. Hence one has $\text{ind-match}(G \setminus v) \geq |(M \setminus \{v, w\}) \setminus \{u, w\}| = |M| = \text{ind-match}(G)$.

(2) : Let M' be a maximal matching of $G \setminus v$ with $M' = \text{min-match}(G \setminus v)$. If $w \notin V(M')$, then $u \notin V(M')$ since u is only adjacent to w. Hence $M' \cup \{u, w\}$ is a matching of $G \setminus v$, but this contradicts the maximality of M'. Thus $w \in V(M')$ and M' is a maximal matching of G. Therefore we have $\text{min-match}(G \setminus v) \geq |M'| = \text{min-match}(G)$.
(3) : We can prove this by replacing "an induced matching" with "a matching" and "ind-match" with "match" in the proof of (1). ☐

2.3. Special families of connected simple graphs $G^{(1)}_{a,b,c}$, $G^{(2)}_{a,b,c,d,e}$ and $G^{(3)}_{a,b,c}$

In this subsection, we introduce three families of connected simple graphs $G^{(1)}_{a,b,c}$, $G^{(2)}_{a,b,c,d,e}$ and $G^{(3)}_{a,b,c}$. These graphs play an important role in the proof of main results.

First, we introduce the graph $G^{(1)}_{a,b,c}$.

$G^{(1)}_{a,b,c}$: Let a, b, c be integers with $a \geq 1$, $a \geq b \geq 0$ and $c \geq 0$ and set

$$X = \{x_1, x_2, \ldots, x_{2a}\},\ Y = \{y_1, y_2, \ldots, y_{2b}\},\ Z = \{z_1, z_2, \ldots, z_c\}.$$

Note that we consider $Y = \emptyset$ if $b = 0$ and $Z = \emptyset$ if $c = 0$. We define the graph $G^{(1)}_{a,b,c}$ as follows; see Figure 2:

- $V\left(G^{(1)}_{a,b,c}\right) = X \cup Y \cup Z$,
- $E\left(G^{(1)}_{a,b,c}\right) = \left\{ \bigcup_{1 \leq i < j \leq 2a} \{x_i, x_j\} \right\} \cup \left\{ \bigcup_{i=1}^{2b} \{x_i, y_i\} \right\} \cup \left\{ \bigcup_{i=1}^{c} \{x_{2a}, z_i\} \right\}$.

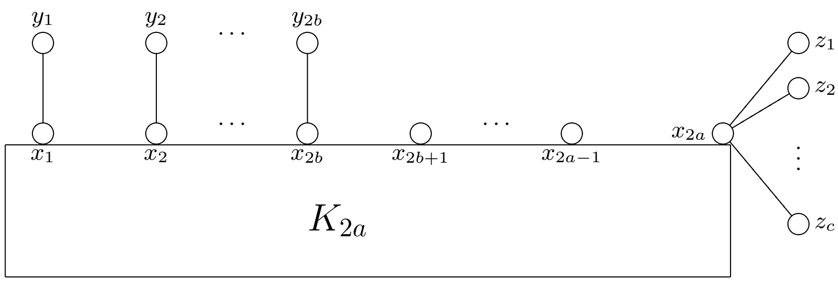

Figure 2. **The graph $G^{(1)}_{a,b,c}$**

Lemma 2.8. *Let $G^{(1)}_{a,b,c}$ be the graph as above. Then we have*

(1) $\left|V\left(G^{(1)}_{a,b,c}\right)\right| = 2a + 2b + c$.

(2) *ind-match* $\left(G^{(1)}_{a,b,c}\right) = 1$.

(3) *min-match* $\left(G^{(1)}_{a,b,c}\right) = a$.

(4) *match* $\left(G^{(1)}_{a,b,c}\right) = a + b$.

Proof. (1) : $\left|V\left(G^{(1)}_{a,b,c}\right)\right| = |X \cup Y \cup Z| = 2a + 2b + c$.
(2) : Since each edge of $G^{(1)}_{a,b,c}$ contains a vertex of the complete subgraph K_{2a}, there is no induced matching M of $G^{(1)}_{a,b,c}$ with $|M| \geq 2$. Hence one has ind-match $\left(G^{(1)}_{a,b,c}\right) = 1$.
(3) : By using Corollary 2.5, it follows that

$$\text{min-match}\left(G^{(1)}_{a,b,c}\right) \geq \text{min-match}\left(\left(G^{(1)}_{a,b,c}\right)_X\right) = \text{min-match}(K_{2a}) = a.$$

Moreover, it also follows that min-match $\left(G^{(1)}_{a,b,c}\right) \leq a$ since $\bigcup_{i=1}^{a}\{x_i, x_{a+i}\}$ is a maximal matching of $\left(G^{(1)}_{a,b,c}\right)$. Thus we have min-match $\left(G^{(1)}_{a,b,c}\right) = a$.
(4) : Let $M = \left\{\bigcup_{i=1}^{2b}\{x_i, y_i\}\right\} \cup \left\{\bigcup_{i=1}^{a-b}\{x_{2b+i}, x_{a+b+i}\}\right\}$. Then one has match $\left(G^{(1)}_{a,b,c}\right) \geq a + b$ since M is a matching of $G^{(1)}_{a,b,c}$ with $|M| = a + b$. If $Z = \emptyset$, then match $\left(G^{(1)}_{a,b,c}\right) = a + b$ holds since M is a perfect matching of $G^{(1)}_{a,b,c}$.

Assume that $Z \neq \emptyset$. Then, by virtue of Proposition 2.1(2) together with Lemma 2.7, it follows that

$$\text{match}\left(G^{(1)}_{a,b,c}\right) = \text{match}\left(G^{(1)}_{a,b,1}\right) \leq \left\lfloor \frac{\left|G^{(1)}_{a,b,1}\right|}{2} \right\rfloor = \left\lfloor \frac{2a + 2b + 1}{2} \right\rfloor = a + b.$$

Hence we have match $\left(G^{(1)}_{a,b,c}\right) = a + b$. □

Next, we introduce the graph $G^{(2)}_{a,b,c,d,e}$.

$\underline{G^{(2)}_{a,b,c,d,e}}$: Let a, b, c, d, e be integers with $a > b \geq 0$, $c \geq 1$, $d, e \geq 0$ and $d + e \geq 1$ and set

$$X = \{x_1, x_2, \ldots, x_{2a}\},\ Y = \{y_1, y_2, \ldots, y_{2b}\},\ Z = \{z_1, z_2, \ldots, z_c\},$$

$$U = \{u_1, u_2, \ldots, u_{2d}\},\ U' = \{u'_1, u'_2, \ldots, u'_{2d}\},\ V = \{v_1, v_2, \ldots, v_{2e}\}.$$

Note that we consider $Y = \emptyset$ if $b = 0$, $U = U' = \emptyset$ if $d = 0$ and $V = \emptyset$ if $e = 0$. We define the graph $G^{(2)}_{a,b,c,d,e}$ as follows; see Figure 3:

- $V\left(G^{(2)}_{a,b,c,d,e}\right) = X \cup Y \cup Z \cup U \cup U' \cup V \cup \{w\}$,
- $E\left(G^{(2)}_{a,b,c,d,e}\right)$

$$= \left\{\bigcup_{1 \leq i < j \leq 2a} \{x_i, x_j\}\right\} \cup \left\{\bigcup_{i=1}^{2b}\{x_i, y_i\}\right\} \cup \left\{\bigcup_{i=1}^{c}\{x_{2a}, z_i\}\right\} \cup \left\{\bigcup_{i=1}^{d}\{u_i, u_{d+i}\}\right\}$$
$$\cup \left\{\bigcup_{i=1}^{d}\{u_i, u'_i\}\right\} \cup \left\{\bigcup_{i=1}^{e}\{v_i, v_{e+i}\}\right\} \cup \{\{w, w'\} \mid w' \in X \cup V \cup U\}.$$

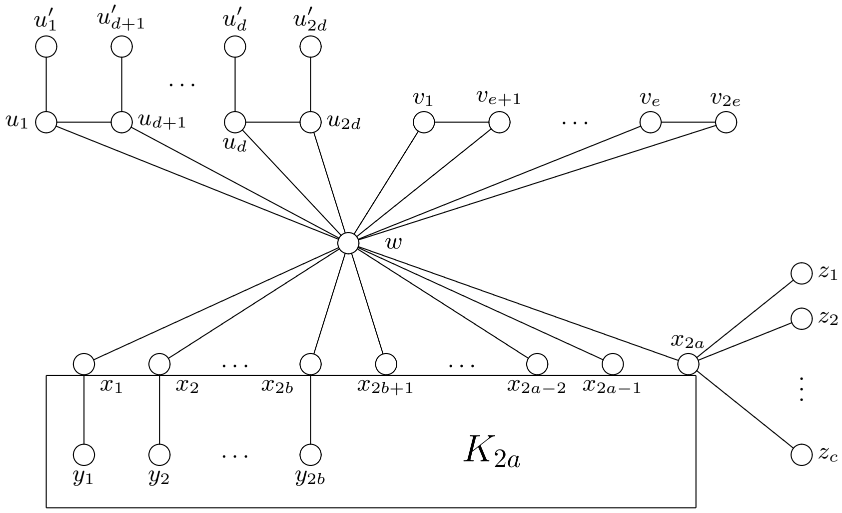

Figure 3. The graph $G^{(2)}_{a,b,c,d,e}$

Lemma 2.9. Let $G^{(2)}_{a,b,c,d,e}$ be the graph as above. Then we have

1. $\left|V\left(G^{(2)}_{a,b,c,d,e}\right)\right| = 2a + 2b + c + 4d + 2e + 1$.
2. ind-match $\left(G^{(2)}_{a,b,c,d,e}\right) = d + e + 1$.
3. min-match $\left(G^{(2)}_{a,b,c,d,e}\right) = a + d + e$.
4. match $\left(G^{(2)}_{a,b,c,d,e}\right) = a + b + 2d + e + 1$.

Proof. (1) : $\left|V\left(G^{(2)}_{a,b,c,d,e}\right)\right| = |X \cup Y \cup Z \cup U \cup U' \cup V \cup \{w\}| = 2a + 2b + c + 4d + 2e + 1$.
To prove (2), (3) and (4), we calculate the induced matching number and the minimum matching number of the induced subgraphs $\left\{G^{(2)}_{a,b,c,d,e}\right\}_{X\cup Y\cup Z}$, $\left\{G^{(2)}_{a,b,c,d,e}\right\}_{U\cup U'}$ and $\left\{G^{(2)}_{a,b,c,d,e}\right\}_{V}$.

- Since $\left\{G^{(2)}_{a,b,c,d,e}\right\}_{X\cup Y\cup Z} = G^{(1)}_{a,b,c}$, by virtue of Lemma 2.8, it follows that
  - ind-match $\left(\left\{G^{(2)}_{a,b,c,d,e}\right\}_{X\cup Y\cup Z}\right) = 1$.
  - min-match $\left(\left\{G^{(2)}_{a,b,c,d,e}\right\}_{X\cup Y\cup Z}\right) = a$.
- For each $1 \le i \le d$, let $U_i = \{u_i, u_{d+i}, u'_i, u'_{d+i}\}$. Then $\left\{G^{(2)}_{a,b,c,d,e}\right\}_{U_i} = P_4$ and

$$\left\{G^{(2)}_{a,b,c,d,e}\right\}_{U\cup U'} = \bigcup_{i=1}^{d} \left\{G^{(2)}_{a,b,c,d,e}\right\}_{U_i} = dP_4,$$

  where P_4 is the path graph with $|V(P_4)| = 4$ and dP_4 is the disjoint union of d copies of P_4. Since ind-match$(P_4) =$ min-match$(P_4) = 1$, by virtue of Lemma 2.6, it follows that

  - ind-match $\left(\left\{G^{(2)}_{a,b,c,d,e}\right\}_{U\cup U'}\right) = \text{ind-match}(dP_4) = d \cdot \text{ind-match}(P_4) = d.$
  - min-match $\left(\left\{G^{(2)}_{a,b,c,d,e}\right\}_{U\cup U'}\right) = \text{min-match}(dP_4) = d \cdot \text{min-match}(P_4) = d.$

- Since $\left\{G^{(2)}_{a,b,c,d,e}\right\}_{V} = eK_2$ and ind-match(K_2) = min-match$(K_2) = 1$, by virtue of Lemma 2.6, it follows that
  - ind-match $\left(\left\{G^{(2)}_{a,b,c,d,e}\right\}_{V}\right) = \text{ind-match}(eK_2) = e \cdot \text{ind-match}(K_2) = e.$
  - min-match $\left(\left\{G^{(2)}_{a,b,c,d,e}\right\}_{V}\right) = \text{min-match}(eK_2) = e \cdot \text{min-match}(K_2) = e.$

Now we are in position to prove (2), (3) and (4).

(2) : Let $S = Y \cup Z \cup U'$. Then S is an independent set of $G^{(2)}_{a,b,c,d,e} \setminus w$ and $G^{(2)}_{a,b,c,d,e}$ is the S-suspension of $G^{(2)}_{a,b,c,d,e} \setminus w$. Since $G^{(2)}_{a,b,c,d,e} \setminus w$ is the disjoint union of $\left\{G^{(2)}_{a,b,c,d,e}\right\}_{X\cup Y\cup Z}$, $\left\{G^{(2)}_{a,b,c,d,e}\right\}_{U\cup U'}$ and $\left\{G^{(2)}_{a,b,c,d,e}\right\}_{V}$, by virtue of Lemmas 2.2, 2.6 and the above calculation, it follows that

$$\begin{aligned}
& \text{ind-match}\left(G^{(2)}_{a,b,c,d,e}\right) \\
= {} & \text{ind-match}\left(G^{(2)}_{a,b,c,d,e} \setminus w\right) \\
= {} & \text{ind-match}\left(\left\{G^{(2)}_{a,b,c,d,e}\right\}_{X\cup Y\cup Z} \cup \left\{G^{(2)}_{a,b,c,d,e}\right\}_{U\cup U'} \cup \left\{G^{(2)}_{a,b,c,d,e}\right\}_{V}\right) \\
= {} & \text{ind-match}\left(\left\{G^{(2)}_{a,b,c,d,e}\right\}_{X\cup Y\cup Z}\right) + \text{ind-match}\left(\left\{G^{(2)}_{a,b,c,d,e}\right\}_{U\cup U'}\right) \\
& + \text{ind-match}\left(\left\{G^{(2)}_{a,b,c,d,e}\right\}_{V}\right) \\
= {} & 1 + d + e.
\end{aligned}$$

(3) : Since $G^{(2)}_{a,b,c,d,e} \setminus w$ is the disjoint union of $\left\{G^{(2)}_{a,b,c,d,e}\right\}_{X\cup Y\cup Z}$, $\left\{G^{(2)}_{a,b,c,d,e}\right\}_{U\cup U'}$ and $\left\{G^{(2)}_{a,b,c,d,e}\right\}_{V}$, by virtue of Corollary 2.5, Lemma 2.6 and the above calculation, it follows that

$$\begin{aligned}
& \text{min-match}\left(G^{(2)}_{a,b,c,d,e}\right) \\
\geq {} & \text{min-match}\left(G^{(2)}_{a,b,c,d,e} \setminus w\right) \\
= {} & \text{min-match}\left(\left\{G^{(2)}_{a,b,c,d,e}\right\}_{X\cup Y\cup Z} \cup \left\{G^{(2)}_{a,b,c,d,e}\right\}_{U\cup U'} \cup \left\{G^{(2)}_{a,b,c,d,e}\right\}_{V}\right) \\
= {} & \text{min-match}\left(\left\{G^{(2)}_{a,b,c,d,e}\right\}_{X\cup Y\cup Z}\right) + \text{min-match}\left(\left\{G^{(2)}_{a,b,c,d,e}\right\}_{U\cup U'}\right) \\
& + \text{min-match}\left(\left\{G^{(2)}_{a,b,c,d,e}\right\}_{V}\right) \\
= {} & a + d + e.
\end{aligned}$$

Next, put $M = \left\{\bigcup_{i=1}^{a}\{x_i, x_{a+i}\}\right\} \cup \left\{\bigcup_{i=1}^{d}\{u_i, u_{d+i}\}\right\} \cup \left\{\bigcup_{i=1}^{e}\{v_i, v_{e+i}\}\right\}$. Then M is a maximal matching with $|M| = a + d + e$ since $V\left(G^{(2)}_{a,b,c,d,e}\right) \setminus V(M) = Y \cup Z \cup U' \cup \{w\}$ is an independent set of $G^{(2)}_{a,b,c,d,e}$. Hence min-match $\left(G^{(2)}_{a,b,c,d,e}\right) \leq |M| = a + d + e$. Thus one has min-match $\left(G^{(2)}_{a,b,c,d,e}\right) = a + d + e$.

(4) : Let

$$\begin{aligned} M' &= \left\{\bigcup_{i=1}^{2b}\{x_i, y_i\}\right\} \cup \left\{\bigcup_{i=1}^{a-b-1}\{x_{2b+i}, x_{a+b-1+i}\}\right\} \cup \left\{\bigcup_{i=1}^{2d}\{u_i, u_i'\}\right\} \\ &\cup \left\{\bigcup_{i=1}^{e}\{v_i, v_{e+i}\}\right\} \cup \{x_{2a-1}, w\} \cup \{x_{2a}, z_1\}. \end{aligned}$$

Then we have match $\left(G^{(2)}_{a,b,c,d,e}\right) \geq |M'| = a+b+2d+e+1$ since M' is a matching of $G^{(2)}_{a,b,c,d,e}$ with $|M| = 2b+(a-b-1)+2d+e+2 = a+b+2d+e+1$. Moreover, by virtue of Proposition 2.1(2) together with Lemma 2.7(3), it follows that

$$\begin{aligned} \text{match}\left(G^{(2)}_{a,b,c,d,e}\right) &= \text{match}\left(G^{(2)}_{a,b,1,d,e}\right) \\ &\leq \left\lfloor \frac{\left|G^{(2)}_{a,b,1,d,e}\right|}{2} \right\rfloor \\ &= \left\lfloor \frac{2a+2b+4d+2e+2}{2} \right\rfloor \\ &= a+b+2d+e+1. \end{aligned}$$

Therefore we have match $\left(G^{(2)}_{a,b,c,d,e}\right) = a+b+2d+e+1$. □

Finally, we introduce the graph $G^{(3)}_{a,b,c}$.

$G^{(3)}_{a,b,c}$: Let a, b, c be integers with $a \geq 1$, $b \geq 0$ and $c \geq 1$ and set

$$X = \{x_1, x_2, \ldots, x_{2a}\},\ Y = \{y_1, y_2, \ldots, y_{2b}\},\ Z = \{z_1, z_2, \ldots, z_c\}.$$

Note that $Y = \emptyset$ if $b = 0$. We define the graph $G^{(3)}_{a,b,c}$ as follows; see Figure 4:

- $V\left(G^{(3)}_{a,b,c}\right) = X \cup Y \cup Z \cup \{v\} \cup \{w\}$,
- $E\left(G^{(3)}_{a,b,c}\right) = \left\{\bigcup_{1\leq i<j\leq 2a}\{x_i, x_j\}\right\} \cup \left\{\bigcup_{i=1}^{b}\{y_i, y_{b+i}\}\right\} \cup \left\{\bigcup_{i=1}^{c}\{v, z_i\}\right\}$
$\cup \{\{w, w'\} \mid w' \in X \cup Y \cup \{v\}\}$.

Lemma 2.10. Let $G^{(3)}_{a,b,c}$ be the graph as above. Then we have

1. $\left|V\left(G^{(3)}_{a,b,c}\right)\right| = 2a+2b+c+2$.
2. ind-match $\left(G^{(3)}_{a,b,c}\right) = b+2$.
3. min-match $\left(G^{(3)}_{a,b,c}\right) =$ match $\left(G^{(3)}_{a,b,c}\right) = a+b+1$.

Proof. (1) : $\left|V\left(G^{(3)}_{a,b,c}\right)\right| = |X \cup Y \cup Z \cup \{v\} \cup \{w\}| = 2a+2b+c+2$.
To prove (2) and (3), we calculate the induced matching number and the minimum matching number of the induced subgraphs $\left\{G^{(3)}_{a,b,c}\right\}_X$, $\left\{G^{(3)}_{a,b,c}\right\}_Y$ and $\left\{G^{(3)}_{a,b,c}\right\}_{Z\cup\{v\}}$.

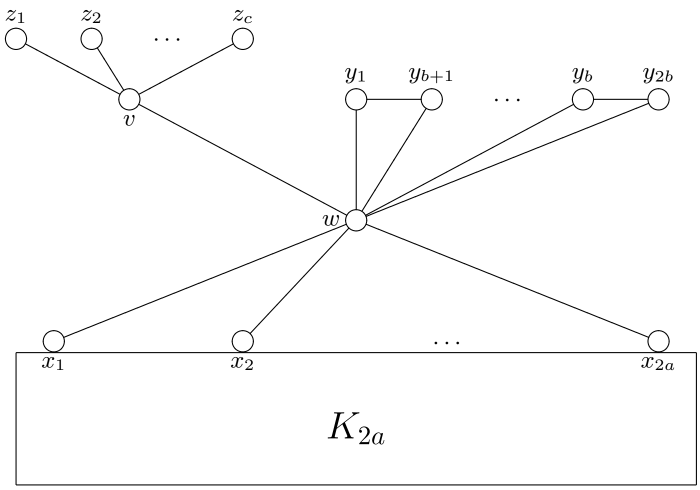

Figure 4. The graph $G^{(3)}_{a,b,c}$

- Since $\left\{G^{(3)}_{a,b,c}\right\}_X = K_{2a}$, it follows that
  - ind-match$\left(\left\{G^{(3)}_{a,b,c}\right\}_X\right) = \text{ind-match}(K_{2a}) = 1$.
  - min-match$\left(\left\{G^{(3)}_{a,b,c}\right\}_X\right) = \text{min-match}(K_{2a}) = a$.
- Since $\left\{G^{(3)}_{a,b,c}\right\}_Y = bK_2$ and ind-match$(K_2) =$ min-match$(K_2) = 1$, by Lemma 2.6, it follows that
  - ind-match$\left(\left\{G^{(3)}_{a,b,c}\right\}_Y\right) = \text{ind-match}(bK_2) = b \cdot \text{ind-match}(K_2) = b$.
  - min-match$\left(\left\{G^{(3)}_{a,b,c}\right\}_Y\right) = \text{min-match}(bK_2) = b \cdot \text{min-match}(K_2) = b$.
- Since $\left\{G^{(3)}_{a,b,c}\right\}_{Z\cup\{v\}}$ is the star graph $K_{1,c}$, it follows that
  - ind-match$\left(\left\{G^{(3)}_{a,b,c}\right\}_{Z\cup\{v\}}\right) = 1$.
  - min-match$\left(\left\{G^{(3)}_{a,b,c}\right\}_{Z\cup\{v\}}\right) = 1$.

Now we are in position to prove (2) and (3).

(2) : Note that Z is an independent set of $G^{(3)}_{a,b,c} \setminus w$ and $G^{(3)}_{a,b,c}$ is the S-suspension of $G^{(3)}_{a,b,c} \setminus w$. Since $G^{(3)}_{a,b,c} \setminus w$ is the disjoint union of $\left\{G^{(3)}_{a,b,c}\right\}_X$, $\left\{G^{(3)}_{a,b,c}\right\}_Y$ and $\left\{G^{(3)}_{a,b,c}\right\}_{Z\cup\{v\}}$, by virtue of Lemmas

2.2, 2.6 and the above calculation, it follows that

$$\begin{aligned}
& \text{ind-match}\left(G^{(3)}_{a,b,c}\right) \\
= \ & \text{ind-match}\left(G^{(3)}_{a,b,c} \setminus w\right) \\
= \ & \text{ind-match}\left(\left\{G^{(3)}_{a,b,c}\right\}_X \cup \left\{G^{(3)}_{a,b,c}\right\}_Y \cup \left\{G^{(3)}_{a,b,c}\right\}_{Z\cup\{v\}}\right) \\
= \ & \text{ind-match}\left(\left\{G^{(3)}_{a,b,c}\right\}_X\right) + \text{ind-match}\left(\left\{G^{(3)}_{a,b,c}\right\}_Y\right) + \text{ind-match}\left(\left\{G^{(3)}_{a,b,c}\right\}_{Z\cup\{v\}}\right) \\
= \ & 1 + b + 1 \\
= \ & b + 2.
\end{aligned}$$

(3) : Since $G^{(3)}_{a,b,c} \setminus w$ is the disjoint union of $\left\{G^{(3)}_{a,b,c}\right\}_X$, $\left\{G^{(3)}_{a,b,c}\right\}_Y$ and $\left\{G^{(3)}_{a,b,c}\right\}_{Z\cup\{v\}}$, by virtue of Corollary 2.5, Lemma 2.6 and the above calculation, it follows that

$$\begin{aligned}
& \text{min-match}\left(G^{(3)}_{a,b,c}\right) \\
\geq \ & \text{min-match}\left(G^{(3)}_{a,b,c} \setminus w\right) \\
= \ & \text{min-match}\left(\left\{G^{(3)}_{a,b,c}\right\}_X \cup \left\{G^{(3)}_{a,b,c}\right\}_Y \cup \left\{G^{(3)}_{a,b,c}\right\}_{Z\cup\{v\}}\right) \\
= \ & \text{min-match}\left(\left\{G^{(3)}_{a,b,c}\right\}_X\right) + \text{min-match}\left(\left\{G^{(3)}_{a,b,c}\right\}_Y\right) + \text{min-match}\left(\left\{G^{(3)}_{a,b,c}\right\}_{Z\cup\{v\}}\right) \\
= \ & a + b + 1.
\end{aligned}$$

Moreover, by Proposition 2.1 and Lemma 2.7, one has

$$\begin{aligned}
\text{min-match}\left(G^{(3)}_{a,b,c}\right) &\leq \text{match}\left(G^{(3)}_{a,b,c}\right) \\
&= \text{match}\left(G^{(3)}_{a,b,1}\right) \\
&\leq \left\lfloor \frac{\left|G^{(3)}_{a,b,1}\right|}{2} \right\rfloor \\
&= \left\lfloor \frac{2a+2b+3}{2} \right\rfloor \\
&= a + b + 1.
\end{aligned}$$

Therefore we have $\text{min-match}\left(G^{(3)}_{a,b,c}\right) = \text{match}\left(G^{(3)}_{a,b,c}\right) = a + b + 1$. □

3. Proof of the first main result

In this section, we give a proof of the first main result as below.

Theorem 3.1. *Let $n \geq 2$ be an integer and set*

$$\begin{aligned}
& \mathbf{Graph}_{\text{ind-match},\text{min-match},\text{match}}(n) \\
= \ & \left\{ (p,q,r) \in \mathbb{N}^3 \ \middle| \ \begin{array}{l} \text{There exists a connected simple graph } G \text{ with } |V(G)| = n \\ \text{and } \text{ind-match}(G) = p, \ \text{min-match}(G) = q, \ \text{match}(G) = r \end{array} \right\}.
\end{aligned}$$

Then we have the following:

(1) *If n is odd, then*

$$\begin{aligned}&\mathbf{Graph}_{\text{ind-match,min-match,match}}(n)\\ =&\left\{(p,q,r)\in\mathbb{N}^3 \;\middle|\; 1\le p\le q\le r\le 2q \;\text{ and }\; r\le\frac{n-1}{2}\right\}.\end{aligned}$$

(2) *If n is even, then*

$$\begin{aligned}&\mathbf{Graph}_{\text{ind-match,min-match,match}}(n)\\ =&\left\{(1,q,r)\in\mathbb{N}^3 \;\middle|\; 1\le q\le r\le 2q \;\text{ and }\; r\le\frac{n}{2}\right\}\\ \cup&\left\{(p,q,r)\in\mathbb{N}^3 \;\middle|\; 2\le p\le q\le r\le 2q,\; r\le\frac{n}{2} \;\text{ and }\; (q,r)\ne\left(\frac{n}{2},\frac{n}{2}\right)\right\}.\end{aligned}$$

Proof. Assume that $(p,q,r)\in \text{Graph}_{\text{ind-match,min-match,match}}(n)$. Then there exists a connected simple graph G such that $|V(G)|=n$, ind-match$(G)=p$, min-match$(G)=q$ and match$(G)=r$.

- If n is odd, then $\left\lfloor\frac{n}{2}\right\rfloor=\frac{n-1}{2}$. Hence, by Proposition 2.1, one has

$$(p,q,r)\in\left\{(p,q,r)\in\mathbb{N}^3 \;\middle|\; 1\le p\le q\le r\le 2q \;\text{ and }\; r\le\frac{n-1}{2}\right\}.$$

- If n is even, then $\left\lfloor\frac{n}{2}\right\rfloor=\frac{n}{2}$. If $p=1$, then we have

$$(1,q,r)\in\left\{(1,q,r)\in\mathbb{N}^3 \;\middle|\; 1\le q\le r\le 2q \text{ and } r\le\frac{n}{2}\right\}$$

  by Proposition 2.1.
  Assume that $p\ge 2$. By virtue of Propositions 2.1 and 2.3, one has

$$(p,q,r)\in\left\{(p,q,r)\in\mathbb{N}^3 \;\middle|\; 2\le p\le q\le r\le 2q,\; r\le\frac{n}{2} \;\text{ and }\; (q,r)\ne\left(\frac{n}{2},\frac{n}{2}\right)\right\}.$$

We show the reverse inclusion. Assume that n is odd and

$$(p,q,r)\in\left\{(p,q,r)\in\mathbb{N}^3 \;\middle|\; 1\le p\le q\le r\le 2q \;\text{ and }\; r\le\frac{n-1}{2}\right\}.$$

- Assume $p=1$. Note that $q\ge 1$, $q\ge r-q\ge 0$ and $n-2r\ge 1$. Let us consider the graph $G^{(1)}_{q,k,n-2(q+k)}$, where $k=r-q$; see Figure 5:
  By virtue of Lemma 2.8, one has
  - $\left|V\left(G^{(1)}_{q,k,n-2(q+k)}\right)\right|=2q+2k+n-2(q+k)=n.$
  - ind-match$\left(G^{(1)}_{q,k,n-2(q+k)}\right)=1.$
  - min-match$\left(G^{(1)}_{q,k,n-2(q+k)}\right)=q.$
  - match$\left(G^{(1)}_{q,k,n-2(q+k)}\right)=q+k=r.$

  Hence we have $(1,q,r)\in\text{Graph}_{\text{ind-match,min-match,match}}(n)$.

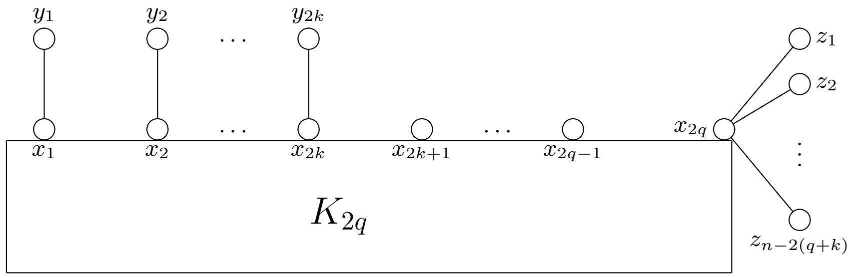

Figure 5. The graph $G^{(1)}_{q,k,n-2(q+k)}$

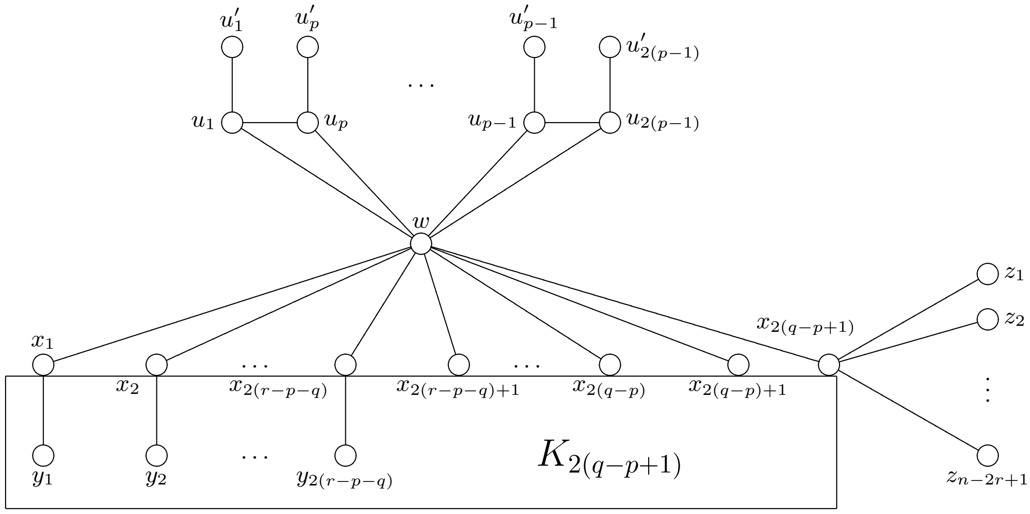

Figure 6. The graph $G^{(2)}_{q-p+1,r-p-q,n-2r+1,p-1,0}$

- Assume that $p \geq 2$ and $p+q-r \leq 0$. Then note that $q-p+1 > r-p-q \geq 0$, $n-2r+1 \geq 1$, $p-1 \geq 1$. Now we consider the graph $G^{(2)}_{q-p+1,r-p-q,n-2r+1,p-1,0}$; see Figure 6. By virtue of Lemma 2.9, one has
  - $\left|V\left(G^{(2)}_{q-p+1,r-p-q,n-2r+1,p-1,0}\right)\right|$
    $= 2(q-p+1) + 2(r-p-q) + (n-2r+1) + 4(p-1) + 1 = n.$
  - ind-match $\left(G^{(2)}_{q-p+1,r-p-q,n-2r+1,p-1,0}\right)$
    $= (p-1) + 1 = p.$
  - min-match $\left(G^{(2)}_{q-p+1,r-p-q,n-2r+1,p-1,0}\right)$
    $= (q-p+1) + (p-1) = q.$
  - match $\left(G^{(2)}_{q-p+1,r-p-q,n-2r+1,p-1,0}\right)$
    $= (q-p+1) + (r-p-q) + 2(p-1) + 1 = r.$

  Thus we have $(p,q,r) \in \text{Graph}_{\text{ind-match,min-match,match}}(n)$.

- Assume that $p \geq 2$, $p+q-r > 0$ and $q < r$. Then note that $q-p+1 > 0$, $n-2r+1 \geq 1$, $r-q-1 \geq 0$ and $p+q-r > 0$. Now we consider the graph $G^{(2)}_{q-p+1,0,n-2r+1,r-q-1,p+q-r}$; see Figure 7. By virtue of Lemma 2.9, one has

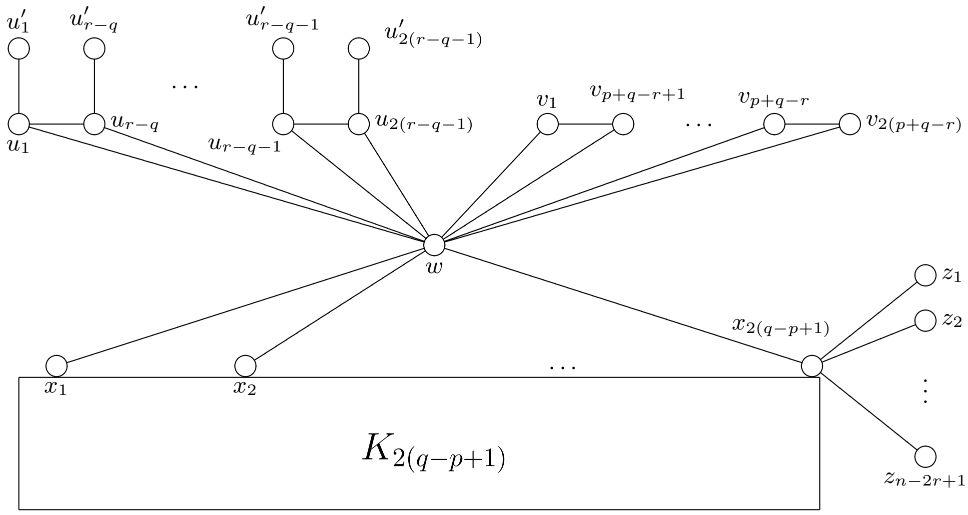

Figure 7. The graph $G^{(2)}_{q-p+1,0,n-2r+1,r-q-1,p+q-r}$

  - $\left|V\left(G^{(2)}_{q-p+1,0,n-2r+1,r-q-1,p+q-r}\right)\right|$
    $= 2(q-p+1)+(n-2r+1)+4(r-q-1)+2(p+q-r)+1 = n.$
  - ind-match $\left(G^{(2)}_{q-p+1,0,n-2r+1,r-q-1,p+q-r}\right)$
    $= (r-q-1)+(p+q-r)+1 = p.$
  - min-match $\left(G^{(2)}_{q-p+1,0,n-2r+1,r-q-1,p+q-r}\right)$
    $= (q-p+1)+(r-q-1)+(p+q-r) = q.$
  - match $\left(G^{(2)}_{q-p+1,0,n-2r+1,r-q-1,p+q-r}\right)$
    $= (q-p+1)+2(r-q-1)+(p+q-r)+1 = r.$

  Thus we have $(p,q,r) \in \text{Graph}_{\text{ind-match,min-match,match}}(n)$.

- Assume that $p \geq 2$ and $q = r$. Note that $q-p+1 \geq 1$, $p-2 \geq 0$ and $n-2q \geq 1$. Now we consider the graph $G^{(3)}_{q-p+1,p-2,n-2q}$; see Figure 8:

  By virtue of Lemma 2.10, one has

  - $\left|V\left(G^{(3)}_{q-p+1,p-2,n-2q}\right)\right| = 2(q-p+1)+2(p-2)+(n-2q)+2 = n.$
  - ind-match $\left(G^{(3)}_{q-p+1,p-2,n-2q}\right) = (p-2)+2 = p.$
  - min-match $\left(G^{(3)}_{q-p+1,p-2,n-2q}\right) =$ match $\left(G^{(3)}_{a,b,c}\right)$
    $= (q-p+1)+(p-2)+1 = q.$

  Thus we have $(p,q,r) \in \text{Graph}_{\text{ind-match,min-match,match}}(n)$.

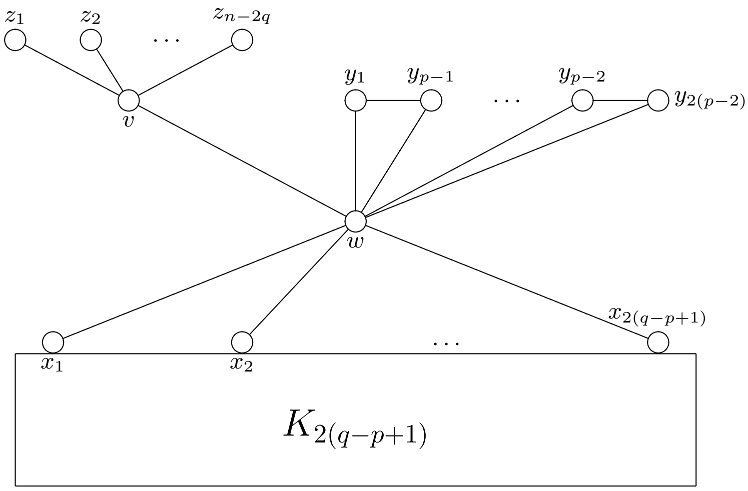

Figure 8. The graph $G^{(3)}_{q-p+1,p-2,n-2q}$

Next, we assume that n is even and

$$(p,q,r) \in \left\{(1,q,r)\in\mathbb{N}^3 \;\middle|\; 1\le q\le r\le 2q \text{ and } r\le \frac{n}{2}\right\} \cup \left\{(p,q,r)\in\mathbb{N}^3 \;\middle|\; 2\le p\le q\le r\le 2q,\ r\le\frac{n}{2} \text{ and } (q,r)\neq\left(\frac{n}{2},\frac{n}{2}\right)\right\}.$$

As in the case that n is odd, we can see that $(p,q,r)\in \mathrm{Graph}_{\text{ind-match,min-match,match}}(n)$ by considering $G^{(1)}_{q,k,n-2(q+k)}$, $G^{(2)}_{q-p+1,r-p-q,n-2r+1,p-1,0}$, $G^{(2)}_{q-p+1,0,n-2r+1,r-q-1,p+q-r}$ and $G^{(3)}_{q-p+1,p-2,n-2q}$.

Therefore, we have the desired conclusion. $\square$

4. The set $\mathrm{Graph}_{\text{reg,min-match,match}}(n)$

In this section, as an application of Theorem 3.1, we determine the possible tuples

$$(\mathrm{reg}(G), \text{min-match}(G), \mathrm{match}(G), |V(G)|)$$

arising from connected simple graphs, where $\mathrm{reg}(G) = \mathrm{reg}(K[V(G)]/I(G))$ denote the *Castelnuovo–Mumford regularity* (regularity for short) of the quotient ring $K[V(G)]/I(G)$ whose definition will be given later.

Let G be a finite simple graph on the vertex set $V(G)=\{x_1,\ldots,x_{|V(G)|}\}$ and $E(G)$ the set of edges of G. Let $K[V(G)] = K\left[x_1,\ldots,x_{|V(G)|}\right]$ be the polynomial ring in $|V(G)|$ variables over a field K. Now we associate with G the quadratic monomial ideal

$$I(G) = (x_ix_j \mid \{x_i,x_j\}\in E(G)) \subset K[V(G)].$$

The ideal $I(G)$ is called the *edge ideal* of G. By setting $\deg x_i = 1$ for all $1\le i\le |V(G)|$, $I(G)$ can be regarded as a homogeneous ideal of $K[V(G)]$. Let $\beta_{i,j}(K[V(G)]/I(G))$ be the (i,j)-th graded Betti number in the minimal graded free resolution of $K[V(G)]/I(G)$. The regularity of $K[V(G)]/I(G)$, denoted by $\mathrm{reg}(G)$, is defined by

$$\mathrm{reg}(G) = \mathrm{reg}(K[V(G)]/I(G)) := \max\{j-i \mid \beta_{i,j}(K[V(G)]/I(G)) \neq 0\}.$$

For more details, see [30, Section 18].

The relationship between graph-theoretical invariants of G and ring-theoretical invariants of the quotient ring $K[V(G)]/I(G)$ has been studied. As previous results,

- In [24, Theorem 1], Hirano and the first-named author determined the possible tuples
$$(\text{ind-match}(G), \text{min-match}(G), \text{match}(G), \dim(G))$$
arising from connected simple graphs, where $\dim(G) = \dim K[V(G)]/I(G)$ denotes the Krull dimension of $K[V(G)]/I(G)$.
- In [23], Hibi et al. proved that
$$\deg(G) + \text{reg}(G) \leq |V(G)|$$
for all simple graph G, where $\deg(G) = \deg h_{K[V(G)]/I(G)}(t)$ denotes the degree of the h-polynomial of $K[V(G)]/I(G)$.
- In [22], Hibi et al. studied the possible tuples $(\text{reg}(G), \deg(G), |V(G)|)$ arising from connected simple graphs G and determined these tuples arising from *Cameron–Walker* graphs, where a finite connected simple graph G is said to be a Cameron–Walker graph if $\text{ind-match}(G) = \text{min-match}(G) = \text{match}(G)$ and if G is neither a star graph nor a star triangle.
- In [19], Hibi et al. studied the possible tuples $(\text{depth}(G), \dim(G), |V(G)|)$ arising from connected simple graphs G and determined these tuples arising from Cameron–Walker graphs. They also determined the possible tuples
$$(\text{depth}(G), \text{reg}(G), \dim(G), \deg(G), |V(G)|)$$
arising from Cameron–Walker graphs, where $\text{depth}(G) = \text{depth}(K[V(G)]/I(G))$ denotes the depth of $K[V(G)]/I(G)$;
- Erey–Hibi determined the possible tuples $(\text{pd}(G), \text{reg}(G), |V(G)|)$ arising from connected *bipartite* graphs, where $\text{pd}(G) = \text{pd}(K[V(G)]/I(G))$ denotes the projective dimension of $K[V(G)]/I(G)$ ([10, Theorem 3.14]). These tuples were also studied in [14].

The second main result is as follows. We determine the possible tuples

$$(\text{reg}(G), \text{min-match}(G), \text{match}(G), |V(G)|)$$

arising from connected simple graphs.

Theorem 4.1. *Let $n \geq 2$ be an integer and set*

$$\begin{aligned}&\mathbf{Graph}_{\text{reg,min-match,match}}(n)\\ =& \left\{ (p', q, r) \in \mathbb{N}^3 \;\middle|\; \begin{array}{c}\text{There exists a connected simple graph } G \text{ with } |V(G)| = n\\ \text{and } \ \text{reg}(G) = p', \ \text{min-match}(G) = q, \ \text{match}(G) = r\end{array} \right\}.\end{aligned}$$

Then one has

$$\mathbf{Graph}_{\text{reg,min-match,match}}(n) = \mathbf{Graph}_{\text{ind-match,min-match,match}}(n).$$

Proof. From [38] and Proposition 2.1(1), we have that

$$\text{reg}(G) \leq \text{min-match}(G) \leq \text{match}(G) \leq 2\text{min-match}(G)$$

holds for all connected graph G. Since the graphs

- $G^{(1)}_{q,k,n-2(q+k)}$,
- $G^{(2)}_{q-p+1,r-p-q,n-2r+1,p-1,0}$,
- $G^{(2)}_{q-p+1,0,n-2r+1,r-q-1,p+q-r}$ and
- $G^{(3)}_{q-p+1,p-2,n-2q}$

which appeared in the proof of Theorem 3.1 are chordal, hence it follows that the regularity of these graphs equal to its induced matching number by virtue of [15, Corollary 6.9]). Moreover, since both of the complements of K_n and $K_{n/2,n/2}$ are chordal, one has $\text{reg}(K_n) = \text{reg}(K_{n/2,n/2}) = 1$ by virtue of Fröberg [11]. Thus, by using Proposition 2.3, we have that there is no connected simple graph G with

$$(\text{reg}(G), \text{min-match}(G), \text{match}(G), |V(G)|) = (p, n/2, n/2, n)$$

for all $p \geq 2$. Therefore we have the desired conclusion. □

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