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Multidecomposition of complete graphs into cycles and claws

Research Article

Panneerselvam Lakshmanan, Ilayaraja Maniprakasam, Muthusamy Appu

Abstract: Let C_n and S_n respectively denote a cycle and star with n edges. Let K_n denote a complete graph on n vertices. In this paper, it is shown that for any non-negative integers α and β and any positive integer $n \geq 6$, there exists a decomposition of K_n into α copies of C_6 and β copies of S_3 if and only if $6\alpha + 3\beta = \binom{n}{2}$, $\beta \neq 1, 2$ when n is odd, and $\beta \geq \lceil \frac{n}{4} \rceil$ when n is even.

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1. Introduction

All graphs considered here are finite. For the standard graph-theoretic terminology the reader is referred to [9].

Let K_k denote a complete graph on k vertices. Let P_{k+1}, C_k and $S_k(\cong K_{1,k})$ respectively denote a path, cycle and star each having k edges. Further, we denote a k-cycle ie., C_k with vertices $1, 2, \ldots, k$ and edges $12, 23, \ldots, k-1k, k1$ by $(1, 2, \ldots, k)$, and the k-star S_k with center vertex 0, end vertices $1, 2, \ldots, k$ and edges $01, 02, \ldots, 0k$ by $(0; 1, 2, \ldots, k)$.When $x, y \in \mathbb{Z}^+$, we define $\lfloor x \rfloor = \max\{y | y \in \mathbb{Z}^+,\ y \leq x\}$ and $\lceil x \rceil = \min\{y | y \in \mathbb{Z}^+,\ y \geq x\}$.

A *decomposition* of a graph G is a partition of G into edge-disjoint subgraphs of G. If the subgraphs in the decomposition are isomorphic to either H_1 or H_2, then it is called an $\{H_1, H_2\}$-*decomposition* of

Panneerselvam Lakshmanan; Department of Mathematics, Faculty of Science and Humanities, SRM Institute of Science and Technology, Ramapuram Campus,Chennai-600089,Tamil Nadu, India.(email:panselneshu@gmail.com).
Ilayaraja Maniprakasam; Department of Mathematics, Sona College of Arts and Science, Salem-5, Tamil Nadu, India. (email: ilayasnow@gmail.com).
Muthusamy Appu (Corresponding Author); Department of Mathematics, Sona College of Technology, Salem–636005, Tamilnadu, India. (email: appumuthusamy@gmail.com).

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G. We say that G has a $\{H_1, H_2\}^{\{\alpha,\beta\}}$*-decomposition* of G if the decomposition contains α copies of H_1 and β copies of H_2 for all possible choices of α and β. Existence of such decomposition of G for $\alpha, \beta \geq 1$ is also called multidecomposition.

Abueida and Daven [1] initiated the study of an $\{H_1, H_2\}$-decomposition of K_n, when H_1 and H_2 are cycles and stars each with 3 edges. In [2, 3], they also proved such decomposition when both the subgraphs contains 4 edges, and $\{C_k, E_2\}$-decomposition of product graphs, when E_2 is a matching of size 2. Moreover, Abueida and O'Neil [5] have proved the existence of $\{C_k, S_{k-1}\}$-decomposition of λK_n, when $k = 3, 4, 5$. Abueida et al. [4] considered the existence of $\{H_1, H_2\}$-decomposition of λK_n, when $\{H_1, H_2\}$ is a graph-pair of orders 4 and 5. Abueida and Lian [6] have obtained necessary conditions and proved that such conditions are also sufficient for $\{C_k, S_k\}$-decomposition of K_n. Beggas et al. [10] proved the existence of $\{C_k, S_k\}$-decomposition of λK_n.

Priyadharsini and Muthusamy [18, 20] have obtained necessary conditions and proved that such conditions are also sufficient for $\{G_n, H_n\}$-factorization of λK_n, $\lambda K_{n,n}$, when G_n, $H_n \in \{C_n, P_n, S_n\}$. The same authors [19] have obtained necessary conditions and proved such conditions are sufficient for the existence of $\{G_n, H_n\}$-decomposition of λK_n.

Lee and Chu [16] have proved the existence of $\{P_{k+1}, S_k\}$-decomposition of $K_{n,n}$. Further, Lee [15] and Lin [17] respectively investigated the decomposition of $K_{m,n}$ and $K_{n,n} - I$ into cycles and stars. Shyu [22–24, 26] considered and proved the existence of $\{P_{k+1}, S_k\}^{\{\alpha,\beta\}}$-decomposition of K_n and $K_{m,n}$ for some values of k. Jeevadoss and Muthusamy [13, 14] have proved the $\{P_{k+1}, C_k\}^{\{\alpha,\beta\}}$-decomposition of $K_{m,n}$ and $\lambda K_{m,n}$ respectively. Fu et al. [11, 12] have obtained necessary conditions and proved that such conditions are also sufficient for $\{C_k, S_3\}^{\{\alpha,\beta\}}$-decomposition of K_n, where $k = 3, 4$.

In [25] Shyu has proved the following:

Theorem 1.1. *[25] Let $\alpha, \beta \in \mathbb{Z}^+ \cup \{0\}$, and $n, l, k \in \mathbb{Z}^+$, and $n \geq$ max $\{\ell, k+1\}$. If K_n has a $\{C_\ell, S_k\}^{\{\alpha,\beta\}}$-decomposition, then $\alpha\ell + \beta k = \binom{n}{2}$, $\beta \neq 1, 2$ when n is odd, and $\beta \geq \lceil \frac{n}{4} \rceil$ when n is even.*

In this paper, we have shown that necessary conditions stated in 1.1 are sufficient for $\ell = 6$ and $k = 3$.

2. Notations and preliminaries

We present some notations and well-known results that are necessary to prove our main results. Let $\{C_\ell, S_k\}$- decomposition of a graph G denotes a decomposition of G into α copies of ℓ- cycles and β copies of k- stars. For a given graph G and for an non-negative integers $\alpha, \beta \geq 0$, define $I(n)$=$\{\beta \mid 2\alpha + \beta = \frac{n(n-1)}{6}, \beta \neq 1, 2$ for odd n, and $\beta \geq \lceil \frac{n}{4} \rceil$ when n is even$\}$. and $M(G)$=$\{\beta \mid G = \alpha C_6 \oplus \beta S_3\}$, i.e., there exit a $\{C_6, S_3\}^{\{\alpha,\beta\}}$ decomposition of G and $\mathbb{G}$ be the set of graphs G for which $M(G) = I(E(G))$. According to the values of n, define the $I(n)$ as follows:

(i) If $n \equiv 0, 4, 6$ or $10 \pmod{12}$, then $I(n) = \left\{\beta \mid \beta = \frac{n(n-1)}{6} - 2i,\ \text{where } 0 \leq i \leq \left\lfloor \frac{n(n-1)}{12} - \frac{n}{8} \right\rfloor \right\}$.

(ii) If $n \equiv 1$ or $9 \pmod{12}$, then $I(n) = \left\{\beta \mid \beta = \frac{n(n-1)}{6} - 2i,\ \text{where } 0 \leq i \leq \left\lfloor \frac{n(n-1)}{12} \right\rfloor \text{ and } \beta \neq 1, 2\right\}$.

(iii) If $n \equiv 3$ or $7 \pmod{12}$, then $I(n) = \left\{\beta \mid \beta = \frac{n(n-1)}{6} - 2i,\ \text{where } 0 \leq i \leq \left\lfloor \frac{n(n-1)-12}{12} \right\rfloor \text{ and } \beta \neq 1, 2\right\}$.

Theorem 2.1. *[7, 21] Let $\ell, n \in \mathbb{Z}^+$. Then K_n has a C_ℓ-decomposition if and only if n is odd, $3 \leq \ell \leq n$ and $n(n-1) \equiv 0 \pmod{2l}$.*

Theorem 2.2. *[28, 29] Let $k, n \in \mathbb{Z}^+$. Then K_n has a S_k-decomposition if and only if $2k \leq n$ and $n(n-1) \equiv 0 \pmod{2k}$.*

Theorem 2.3. *[27] Let ℓ, m and $n \in \mathbb{Z}^+$. Then $K_{m,n}$ has a $C_{2\ell}$-decomposition if and only if m and n are even, $m, n \geq \ell \geq 2$ and $mn \equiv 0 \pmod{2\ell}$.*

Theorem 2.4. *[29] Let k, m and $n \in \mathbb{Z}^+$ with $m \leq n$. There exists an S_k-decomposition of $K_{m,n}$ if and only if either (i) $k \leq m$ and $mn \equiv 0 \pmod{k}$ or (ii) $m < k \leq n$ and $n \equiv 0 \pmod{k}$.*

Theorem 2.5. *[8] Let G be a forest in K_n with $|E(G)| \geq 1$. There exists a 6-cycle system of $K_n \setminus G$ if and only if (i) all vertices in G have odd degree (ii) $|E(K_n \setminus G)|$ is divisible by 6, and (iii) n is even.*

3. Base constructions

In this section, we provide some useful lemmas which are required in proving our main result.

Lemma 3.1. *There exists a $\{C_6, S_3\}^{\{\alpha,\beta\}}$-decomposition of $K_{4,6}$ with $\beta \in \{0, 8\}$*

Proof. (i) $\alpha = 4$ and $\beta = 0$: By 2.3, there exist four copies of C_6.
(ii)$\alpha = 0$ and $\beta = 8$: By 2.4, there exist eight copies of S_3.
Hence, $M(K_{4,6}) = \{0, 8\}$. □

Lemma 3.2. *There exists a $\{C_6, S_3\}^{\{\alpha,\beta\}}$-decomposition of $M(K_{6,6})$ with $\beta \in \{0, 4, 6, 8, 10, 12\}$*

Proof. Let $V(K_{6,6}) = (\{1, \ldots, 6\}, \{7, \ldots, 12\})$.
(i) $\alpha = 6$ and $\beta = 0$: By 2.3, we get $6C_6$.
(ii)$\alpha = 4$ and $\beta = 4$: The required cycles and stars are
$(1, 8, 4, 7, 3, 10), (1, 11, 4, 9, 2, 12), (2, 10, 4, 12, 3, 11), (1, 7, 2, 8, 3, 9), (5; 7, 8, 9)$
$(5; 10, 11, 12), (6; 7, 8, 9), (6; 10, 11, 12)$.
(iii)$\alpha = 3$ and $\beta = 6$:The required cycles and stars are
$(1, 8, 4, 7, 3, 10), (2, 10, 4, 12, 3, 11), (1, 7, 2, 8, 3, 9), (5; 7, 8, 10), (6; 7, 8, 9)$
$(6; 10, 11, 12), (9; 2, 4, 5), (11; 1, 4, 5), (12; 1, 2, 5)$.
(iv)$\alpha = 2$ and $\beta = 8$:The required cycles and stars are
$(1, 7, 3, 11, 4, 10), (1, 8, 2, 12, 3, 9), (2; 9, 10, 11), (6; 7, 8, 12), (7; 2, 4, 5)$,
$(8; 3, 4, 5), (9; 4, 5, 6), (10; 3, 5, 6), (11; 1, 5, 6), (12; 1, 4, 5)$.
(v)$\alpha = 1$ and $\beta = 10$:The required cycles and stars are
$(1, 7, 3, 9, 2, 8), (1; 9, 10, 11), (2; 10, 11, 12), (3; 10, 11, 12), (6; 7, 8, 12)$,
$(7; 2, 4, 5), (8; 3, 4, 5), (9; 4, 5, 6), (10; 4, 5, 6), (11; 4, 5, 6), (12; 1, 4, 5)$.
(vi)$\alpha = 0$ and $\beta = 12$: By 2.4, we get the required 12 copies S_3.
Hence, $M(K_{6,6}) = \{0, 4, 6, 8, 10, 12\}$. □

Lemma 3.3. *$M(K_6) = I(6)$.*

Proof. Since $I(6) = \{3, 5\}$. Let $V(K_6) = \{1, 2, \cdots, 6\}$
(i) $\alpha = 1$ and $\beta = 3$:
Let $C_6^1 = (1, 2, 3, 4, 5, 6)$ be a 6-cycle in K_6. Then the graph $H = K_6 \setminus C_6^1$ is 3-regular. Let $S_3^1 = (1; 3, 4, 5)$ be a star in H. It is easy to see that the remaining graph $H \setminus S_3^1$ cannot be decomposed into $2S_3$.
(ii) $\alpha = 0$ and $\beta = 5$: By 2.2, we get five copies of S_3.
Hence, $M(K_6) = I(6) = \{5\}$. □

Lemma 3.4. *$M(K_7) = I(7)$.*

Proof. Since $I(7) = \{3, 5, 7\}$. Let $V(K_7) = \{1, \ldots, 7\}$. We complete the proof in three cases.
(i) $\alpha = 2$ and $\beta = 3$: Let D be an arbitrary C_6 and S_3 decomposition of K_7. Suppose that

C_6^1=(1, 2, 3, 4, 5, 6) and C_6^2=(2, 4, 6, 3, 5, 7) are 2 C_6 in D. By our assumption the graph $H = K_7 \setminus E(C_6^1 \cup C_6^2)$ has an S_3-decomposition. Let $d(x)$ be the degree of a vertex x in H. Then $d(1) = d(7) = 4$ and $d(2) = d(3) = d(4) = d(5) = d(6) = 2$. It follows that only the vertices 1 and 7 must be a center vertex of stars in D. Hence the graph H cannot be decomposed into $3S_3$.
(ii) $\alpha = 1$ and $\beta = 5$: The required cycles and stars are
$(1, 2, 3, 4, 5, 6), (1; 3, 4, 5), (2; 5, 6, 7), (3; 5, 6, 7), (4; 2, 6, 7), (7; 1, 5, 6)$.
(iii) $\alpha = 0$ and $\beta = 7$: By 2.2, we get $7S_3$.
Hence, $M(K_7) = I(7) = \{3, 5, 7\}$. □

Lemma 3.5. $M(K_9) = I(9)$.

Proof. Since $I(9) = \{0, 4, 6, 8, 10, 12\}$. Let $V(K_9) = \{1, \dots, 9\}$.We complete the proof in six cases as follows
(i) $\alpha = 6$ and $\beta = 0$: By 2.1, we get $6C_6$.
(ii) $\alpha = 4$ and $\beta = 4$: The required cycles and stars are
$(3, 6, 9, 2, 4, 8), (1, 5, 8, 2, 7, 6), (3, 5, 6, 8, 1, 9), (1, 3, 4, 9, 8, 7), (2; 1, 3, 6), (4; 1, 6, 7), (5; 2, 4, 9), (7; 3, 5, 9)$.
(iii) $\alpha = 3$ and $\beta = 6$: The required cycles and stars are
$(1, 2, 3, 4, 5, 6), (1, 4, 6, 7, 8, 9), (2, 8, 3, 7, 4, 9), (1; 5, 7, 8), (2; 4, 5, 6), (3; 1, 5, 6), (7; 2, 5, 9), (8; 4, 5, 6), (9; 3, 5, 6)$.
(iv) $\alpha = 2$ and $\beta = 8$: The required cycles and stars are
$(1, 3, 9, 4, 8, 5), (9, 2, 8, 3, 7, 5), (1; 2, 6, 7), (2; 3, 4, 5), (3; 4, 5, 6), (4; 1, 5, 7), (6; 2, 4, 5), (7; 2, 6, 9), (8; 1, 6, 7), (9; 1, 6, 8)$.
(v) $\alpha = 1$ and $\beta = 10$: The required cycles and stars are
$(1, 2, 3, 4, 5, 9), (1; 3, 4, 5), (1; 6, 7, 8), (2; 4, 5, 6), (2; 7, 8, 9), (3; 5, 6, 7), (6; 4, 5, 7), (7; 4, 5, 9), (8; 3, 7, 9), (8; 4, 5, 6), (9; 3, 4, 6)$.
(vi) $\alpha = 0$ and $\beta = 12$: By 2.2, we get $12S_3$.
Hence, $M(K_9) = I(9) = \{0, 4, 6, 8, 10, 12\}$. □

Lemma 3.6. $M(K_{10}) = I(10)$.

Proof. Since $I(10) = \{2x + 1 | 1 \le x \le 7\}$. Let $K_{10} = K_9 \oplus K_{1,9}$. Then by 3.5 and 2.4, we have $M(K_{10}) \supseteq \{0, 4, 6, 8, 10, 12\} + \{3\} = \{3, 7, 9, 11, 13, 15\}$. Let $V(K_{10}) = \{1, \dots, 10\}$. Then the required cycles and stars for the case $(\alpha, \beta) = (5, 5)$ are $(2, 4, 10, 7, 5, 9), (2, 3, 4, 6, 9, 1), (2, 5, 8, 3, 1, 7), (10, 3, 6, 8, 7, 9), (10, 6, 5, 1, 4, 8), (1; 6, 8, 10), (2; 6, 8, 10), (5; 3, 4, 10), (7; 3, 4, 6), (9; 3, 4, 8)$. Hence, $M(K_{10}) = I(10) = \{2x + 1 | 1 \le x \le 7\}$. □

Lemma 3.7. $M(K_{12}) = I(12)$.

Proof. Since $I(12) = \{2x | 2 \le x \le 11\} \cup \{0\}$. We write K_{12}=$K_9 \oplus K_3 \oplus K_{3,9}$=$K_9 \oplus (K_3 \cup K_{3,9}) = K_9 \oplus H$, where $H = K_3 \cup K_{3,9}$. Let $V(H) = \{1, \dots, 12\}$. Then the decomposition of H for $(\alpha, \beta) = (2, 6)$ is as follows: $(12, 2, 10, 4, 11, 3), (12, 5, 11, 9, 10, 8)$,
$(1; 10, 11, 12), (6; 10, 11, 12), (7; 10, 11, 12), (10; 3, 5, 11), (11; 2, 8, 12), (12; 4, 9, 10)$. Thus $\{6\} \subseteq M(H)$. From the above case and by Lemma 3.5, we get $(\alpha, \beta) = \{(8, 6), (5, 12)\}$. Thus $\{6, 12\} \subseteq M(K_{12})$. To show $\{10, 14, 16, 18, 20, 22\} \subseteq M(K_{12})$, we let $K_{12} = 2K_6 \oplus K_{6,6}$ and apply Lemmas 3.2 and 3.3 we get $\{2x | 3 \le x \le 11 \ \& \ x \ne 4\} \subseteq M(K_{12})$. We complete the remaining proof in three cases.
(i) $\alpha = 11$ and $\beta = 0$:
By 2.1, K_{12} has an $11C_3$-decomposition
(ii) $\alpha = 9$ and $\beta = 4$: The required cycles and stars are
$(1, 3, 12, 4, 11, 5), (1, 6, 11, 3, 9, 2), (1, 11, 7, 4, 9, 10), (1, 9, 7, 6, 8, 12), (2, 10, 8, 7, 5, 3), (2, 11, 10, 7, 12, 6), (2, 12, 5, 6, 10, 4), (3, 10, 5, 9, 8, 4), (4, 6, 9, 11, 8, 5), (1; 4, 7, 8), (2; 5, 7, 8), (3; 6, 7, 8), (12; 9, 10, 11)$.
(iii) $\alpha = 7$ and $\beta = 8$: The required cycles and stars are
$(1, 6, 2, 9, 11, 7), (1, 5, 2, 7, 3, 10), (1, 3, 9, 5, 10, 11), (9, 10, 2, 1, 12, 7), (10, 7, 6, 11, 2, 8), (10, 4, 7, 8, 5, 6), (12, 2, 3, 8, 1, 9), (3; 4, 6, 11), (4; 1, 2, 8)$,

$(4;5,9,11),(5;3,7,11),(6;4,9,12),(8;6,9,11),(12;3,4,5),(12;8,10,11)$.
Therefore $M(K_{12}) = I(12) = \{2x|2 \leq x \leq 11\} \cup \{0\}$. □

Lemma 3.8. $M(K_{13}) = I(13)$.

Proof. Since $I(13) = \{2x|2 \leq x \leq 13\}$. We can write $K_{13} = K_{12} \oplus K_1 \oplus K_{1,3}$. Then by Lemma 3.7 and 2.4, we have $\{2x|4 \leq x \leq 13\} \in M(K_{13})$. Let $V(H) = \{1,\ldots,13\}$. Then $(\alpha,\beta) = \{(5,4),(4,6),(13,0)\}$ and the decomposition of H is given in cases as follows:
(i) $\alpha = 13$ and $\beta = 0$:
By 2.1, K_{13} has $13C_3$-decomposition
(ii) $\alpha = 5$ and $\beta = 4$: The required cycles and stars are
$(13,1,12,2,11,3),(13,2,10,3,12,4),(13,5,10,4,11,8),(13,6,12,5,11,7)$,
$(12,7,10,6,11,9),(10;1,8,9),(11;1,10,12),(12;8,10,13),(13;9,10,11)$.
(iii)The decomposition of K_{13} as $K_9 \oplus K_4 \oplus K_{4,9} = K_9 \oplus H$, where $H = K_4 \oplus K_{4,9}$. $\alpha = 4$ and $\beta = 6$: The required cycles and stars are
$(13,1,12,2,11,3),(13,2,10,3,12,4),(13,5,10,4,11,8),(13,6,12,5,11,7),(10;7,8,9)$,
$(10;1,6,11),(12;9,10,11),(11;1,6,9),(12;7,8,13),(13;9,10,11)$. Thus $\{4,6\} \in M(H)$. From the above case and by 3.5, we get $(\alpha,\beta) = \{(5,4),(4,6)\} + (6,0) = \{(11,4),(10,6)\}$. Hence, $\{4,6\} \in M(K_{13})$. Therefore $M(K_{13}) = I(13) = \{2x|2 \leq x \leq 13\} \cup \{0\}$. □

Lemma 3.9. $M(K_{15}) = I(15)$.

Proof. Since $I(15) = \{2x+1|1 \leq x \leq 17\}$. We write $K_{15} = K_9 \oplus K_7 \oplus K_{8,6} = K_9 \oplus K_7 \oplus 2K_{4,6}$. Then $M(K_{15}) \supseteq M(K_9) + M(K_7) + 2M(K_{4,6})$=$\{0,4,6,8,10,12\} + \{5,7\} + 2\{0,8\}$=$\{2x+1|2 \leq x \leq 17\}$. We can write $K_{15} = K_9 \oplus K_6 \oplus K_{9,6}$=$K_9 \oplus H$, where $H = (K_6 \cup K_{9,6})$. Let $V(H) = \{1,\ldots,15\}$. Then $(\alpha,\beta) = (10,3)$ and the decomposition of H is given as follows:
$(15,1,14,2,13,3),(15,2,12,1,13,4),(15,5,14,3,12,6),(15,7,14,4,12,8)$,
$(15,9,14,6,13,10),(15,11,1,10,5,12),(14,11,2,10,7,13),(14,10,6,11,7,12)$,
$(14,15,13,12,11,8),(11,3,10,8,13,5),(9;11,12,13),(10;4,9,12),(11;4,10,13)$. Thus $\{3\} \in M(H)$. From the above case and by Lemma 3.5, we get $(\alpha,\beta) = (10,3) + (6,0) = (16,3)$. Thus $\{3\} \in M(K_{15})$. Therefore $M(K_{15}) = I(15) = \{2x|1 \leq x \leq 17\}$. □

Lemma 3.10. $M(K_{16}) = I(16)$.

Proof. Since $I(16) = \{2x|2 \leq x \leq 20\}$. Since $K_{16} = K_{10} \oplus K_6 \oplus K_{4,6} \oplus K_{6,6}$. Then $M(K_{16}) \supseteq M(K_{10}) + M(K_6) + M(K_{4,6}) + M(K_{6,6})$=$\{2x+1|1 \leq x \leq 7\} + \{5\} + \{0,8\} + \{0,4,6,8,10,12\}$=$\{2x|4 \leq x \leq 20\}$. Let $V(K_{16}) = \{1,\ldots,16\}$. Then $(\alpha,\beta) = \{(18,4),(17,6)\}$ and we decompose K_{16} in three cases as follows:
(i) $\alpha = 20$ and $\beta = 0$:
By 2.1, K_{16} has $20C_3$-decomposition.
(ii) $\alpha = 18$ and $\beta = 4$: Let $S_3^1 = (13;10,11,12), S_3^2 = (14;7,8,9), S_3^3 = (15;4,5,6)$ and $S_3^4 = (16;1,2,3)$ are $4S_3$'s in K_{16}. Let $H = \cup_{i=1}^4 S_3^i$ is a forest. Then by 2.5, the graph $K_{16} \setminus H$ can be decomposed into $18C_6$. Thus $\{4\} \in M(K_{16})$.
(iii) $\alpha = 17$ and $\beta = 6$: Let $K_{16} = K_{13} \oplus K_3 \oplus K_{13,3}$=$K_{13} \oplus H$, where $H = (K_3 \oplus K_{13,3})$. Then $(\alpha,\beta) = (4,6)$, we can decompose H into $4C_6$ and $6S_3$ as follows:
$(16,1,15,3,14,4),(16,2,15,5,14,6)$,
$(16,7,15,8,14,9),(16,10,15,11,14,12)$ $(14;1,2,7),(14;10,13,15),(15;4,6,16),(15;9,12,13)$,
$(16;3,5,8),(16;11,13,14)$. From the above case and by 1.1, we get $\{6\} \in M(H)$ and $\{0\} \in M(K_{13})$ respectively. Hence, $\{6\} \in M(K_{16})$. Therefore $M(K_{16}) = I(16) = \{2x|2 \leq x \leq 20\} \cup \{0\}$. □

Lemma 3.11. $M(K_{18}) = I(18)$.

Proof. Since $I(18) = \{2x+1|2 \leq x \leq 25\}$. We can write $K_{18} = K_{12} \oplus K_6 \oplus 2K_{6,6}$. Then $M(K_{18}) \supseteq M(K_{12}) + M(K_6) + 2M(K_{6,6})$=$\{2x|2 \leq x \leq 11\} + \{5\} + 2\{0,4,6,8,10,12\}$=$\{2x+1|4 \leq x \leq 25\}$, by Lemmas 3.2, 3.3 and 3.7. Let $V(K_{16}) = \{1,\ldots,16\}$. Then $(\alpha,\beta) = \{(23,5),(22,7)\}$. We decompose K_{18} as follows: (i) Let $S_3^1 = (16;7,8,9), S_3^2 = (16;10,11,12), S_3^3 = (16;13,14,15), S_3^4 = (17;4,5,6)$ and $S_3^5 = (18;1,2,3)$ are $5S_3$'s in K_{18}. Let $H_1 = \cup_{i=1}^5 S_3^i$ is a forest. Then by 2.5, the graph $K_{18} \setminus H_1$

can be decomposed into $23C_6$. Thus $\{5\} \in M(K_{16})$. (ii) Let $S_3^1 = (1; 2, 3, 4), S_3^2 = (1; 14, 15, 16), S_3^3 = (16; 13, 14, 15), S_3^4 = (17; 8, 9, 10), S_3^5 = (17; 11, 12, 13), S_3^6 = (17; 14, 15, 16)$ and $S_3^7 = (18; 5, 6, 7)$ are $7S_3$'s in K_{18}. Let $H_2 = \cup_{i=1}^{7} S_3^i$ is a forest. Then by 2.5, the graph $K_{18} \setminus H_2$ can be decomposed into $22C_6$. Thus $\{7\} \in M(K_{16})$. Therefore $M(K_{18}) = I(18) = \{2x + 1 | 2 \le x \le 25\}$. □

4. Main results

In this section, we prove that K_n can be decomposed into α copies of C_6 and β copies of S_3 for all positive integer $n \ge 6$, where $6\alpha + 3\beta = \binom{n}{2}$, $\beta \ne 1, 2$ when n is odd, and $\beta \ge \lceil \frac{n}{4} \rceil$ when n is even.

Lemma 4.1. *$M(K_{12r}) = I(12r)$, where $r \in \mathbb{Z}^+$.*

Proof. Necessity: The condition $M(K_{12r}) \subseteq I(12r)$ is trivial, by 1.1. Sufficiency: We have to prove that $M(K_{12r}) \supseteq I(12r)$. The proof is by induction on r. If $r = 1$, then $M(K_{12}) = I(12)$, by Lemma 3.7. We write $K_{12r+12} = K_{12r} \oplus K_{12} \oplus K_{12r,12} = K_{12r} \oplus K_{12} \oplus (6r)K_{4,6}$. From the definition of $I(n)$, we have

$$\begin{aligned}
I(24s) &= \left\{\beta \middle| \beta = \frac{(24s)(24s-1)}{6} - 2i,\ 0 \le i \le \left\lfloor \frac{(24s)(24s-1)}{12} - 3s \right\rfloor \right\}, \\
&= \{2x | 3s \le x \le 2(24s^2 - s)\}. \\
I(24s+12) &= \left\{\beta \middle| \beta = \frac{(24s+12)(24s+11)}{6} - 2i,\ 0 \le i \le \left\lfloor \frac{(24s+12)(24s+11)}{12} - \frac{(24s+12)}{8} \right\rfloor \right\}, \\
&= \left\{2x + 2 \middle| \left\lceil \frac{6s+1}{2} \right\rceil \le x \le 2(24s^2 + 23s + 5)\right\}. \\
I(24s+24) &= \left\{\beta \middle| \beta = \frac{(24s+24)(24s+23)}{6} - 2i,\ 0 \le i \le \left\lfloor \frac{(24s+24)(24s+23)}{12} - (3s+3) \right\rfloor \right\}, \\
&= \left\{2x \middle| \left\lceil \frac{6s+5}{2} \right\rceil \le x \le 2(24s^2 + 47s + 23)\right\}.
\end{aligned}$$

Case 1. If $r = 2s$, then $K_{24s+12} = K_{24s} \oplus K_{12} \oplus (12s)K_{4,6}$. By the induction hypothesis, and Lemmas 3.1, 3.7, we have $M(K_{24s+12}) \supseteq M(K_{24s}) + M(K_{12}) + (12s)M(K_{4,6}) = \{2x | 3s \le x \le 2(24s^2 - s)\} + \{2x | 2 \le x \le 11\} \cup \{0\} + (12s)\{0, 8\} = \left\{2y + 2 \middle| \left\lceil \frac{6s+1}{2} \right\rceil \le y \le 2(24s^2 + 23s + 5)\right\}$. Therefore $M(K_{24s+12}) = I(24s + 12)$.
Case 2. If $r = 2s + 1$, then $K_{24s+24} = K_{24s+12} \oplus K_{12} \oplus (12s + 6)K_{4,6}$. By Case 1, and Lemmas 3.1, 3.7, we have $M(K_{24s+24}) \supseteq M(K_{24s+12}) + M(K_{12}) + (12s + 6)M(K_{4,6}) = \left\{2x + 2 \middle| \left\lceil \frac{6s+1}{2} \right\rceil \le x \le 2(24s^2 + 23s + 5)\right\} + \{2x | 2 \le x \le 11\} + (12s + 6)\{0, 8\} = \left\{2y \middle| \left\lceil \frac{6s+5}{2} \right\rceil \le y \le 2(24s^2 + 47s + 23)\right\}$. Therefore $M(K_{24s+24}) = I(24s + 24)$. Thus $M(K_{12r}) = I(12r)$ for each $r \in \mathbb{Z}^+$. □

Lemma 4.2. *$M(K_{12r+1}) = I(12r + 1)$, where $r \in \mathbb{Z}^+$.*

Proof. Necessity: The condition $M(K_{12r+1}) \subseteq I(12r + 1)$ is trivial, by 1.1. Sufficiency: We have to prove that $M(K_{12r+1}) \supseteq I(12r + 1)$. The proof is by induction on r. If $r = 1$, then $M(K_{13}) = I(13)$, by Lemma 3.8. We can write $K_{12r+13} = K_{12r+1} \oplus K_{13} \oplus K_{12r,12} = K_{12r+1} \oplus K_{13} \oplus (6r)K_{4,6}$. From the definition of $I(n)$, we have

$$\begin{aligned}
I(24s+1) &= \left\{\beta \middle| \beta = \frac{(24s+1)(24s)}{6} - 2i,\ 0 \le i \le \left\lfloor \frac{(24s+1)(24s)}{12} \right\rfloor,\ \beta \ne 1, 2\right\}, \\
&= \{2x | 0 \le x \le 2(24s^2 + s), x \ne 1\}. \\
I(24s+13) &= \left\{\beta \middle| \beta = \frac{(24s+13)(24s+12)}{6} - 2i,\ 0 \le i \le \left\lfloor \frac{(24s+13)(24s+12)}{12} \right\rfloor,\ \beta \ne 1, 2\right\}, \\
&= \{2x | 0 \le x \le 2(24s^2 + 25s + 6) + 1, x \ne 1\}. \\
I(24s+25) &= \left\{\beta \middle| \beta = \frac{(24s+25)(24s+24)}{6} - 2i,\ 0 \le i \le \left\lfloor \frac{(24s+25)(24s+24)}{12} \right\rfloor,\ \beta \ne 1, 2\right\}, \\
&= \{2x | 0 \le x \le 2(24s^2 + 49s + 25), x \ne 1\}.
\end{aligned}$$

Case 1. If $r = 2s$, then $K_{24s+13} = K_{24s+1} \oplus K_{13} \oplus (12s)K_{4,6}$. By the induction hypothesis, and Lemmas

3.1, 3.8, we have $M(K_{24s+13}) \supseteq M(K_{24s+1}) + M(K_{13}) + (12s)M(K_{4,6})$= $\{2x|0 \leq x \leq 2(24s^2+s), x \neq 1\} + \{2x|2 \leq x \leq 13\} \cup \{0\} + (12s)\{0,8\} = \{2y|0 \leq y \leq 2(24s^2+25s+6)+1, y \neq 1\}$. Therefore $M(K_{24s+13}) = I(24s+13)$.

Case 2. If $r = 2s+1$, then $K_{24s+25} = K_{24s+13} \oplus K_{13} \oplus (12s+6)K_{4,6}$. By Case 1, and Lemmas 3.1, 3.8, we have $M(K_{24s+25}) \supseteq M(K_{24s+13}) + M(K_{13}) + (12s+6)M(K_{4,6})$=$\{2x|0 \leq x \leq 2(24s^2+25s+6)+1, x \neq 1\} + \{2x|2 \leq x \leq 13\} + (12s+6)\{0,8\}$=$\{2y|0 \leq y \leq 2(24s^2+49s+25), y \neq 1\}$. Therefore $M(K_{24s+25}) = I(24s+25)$. Thus $M(K_{12r+1}) = I(12r+1)$ for each $r \in \mathbb{Z}^+$. □

Lemma 4.3. *$M(K_{12r+3}) = I(12r+3)$, where $r \in \mathbb{Z}^+$.*

Proof. Necessity: The condition $M(K_{12r+3}) \subseteq I(12r+3)$ is trivial, by 1.1. Sufficiency: We have to prove $M(K_{12r+3}) \supseteq I(12r+3)$. The proof is by induction on r. If $r = 1$, then $M(K_{15}) = I(15)$, by Lemma 3.9. We write $K_{12r+3} = K_{12r-5} \oplus K_9 \oplus 2(2r-1)K_{4,6}$. From the definition of $I(n)$, we have

$$\begin{aligned}
I(24s-5) &= \left\{\beta \Big| \beta = \frac{(24s-5)(24s-6)}{6} - 2i,\ 0 \leq i \leq \left\lfloor \frac{(24s-5)(24s-6)-12}{12} \right\rfloor,\ \beta \neq 1,2 \right\}, \\
&= \{2x+1|1 \leq x \leq 2(24s^2-11s+1)\}. \\
I(24s+3) &= \left\{\beta \Big| \beta = \frac{(24s+3)(24s+2)}{6} - 2i,\ 0 \leq i \leq \left\lfloor \frac{(24s+3)(24s+2)-12}{12} \right\rfloor,\ \beta \neq 1,2 \right\}, \\
&= \{2x+1|1 \leq x \leq 2(24s^2+5s)\}. \\
I(24s+7) &= \left\{\beta \Big| \beta = \frac{(24s+7)(24s+6)}{6} - 2i,\ 0 \leq i \leq \left\lfloor \frac{(24s+7)(24s+6)-12}{12} \right\rfloor,\ \beta \neq 1,2 \right\}, \\
&= \{2x+3|0 \leq x \leq 2(24s^2+13s+1)\}. \\
I(24s+15) &= \left\{\beta \Big| \beta = \frac{(24s+15)(24s+14)}{6} - 2i,\ 0 \leq i \leq \left\lfloor \frac{(24s+15)(24s+14)-12}{12} \right\rfloor,\ \beta \neq 1,2 \right\}, \\
&= \{2x+3|0 \leq x \leq 2(24s^2+29s+8)\}.
\end{aligned}$$

Case 1. If $r = 2s$, then $K_{24s+3} = K_{24s-9} \oplus K_{13} \oplus (12s-4)K_{4,6}$. By the induction hypothesis, and by Lemmas 3.1 and 3.5, we have $M(K_{24s+3}) \supseteq M(K_{12r-5}) \oplus M(K_9) \oplus 2(2r-1)M(K_{4,6})$.= $\{2x+1|1 \leq x \leq 2(24s^2-11s+1)\} + \{2x|2 \leq x \leq 13\} + (12s-4)\{0,8\} = \{2y+1|1 \leq y \leq 2(24s^2+5s)\}$. Therefore, $M(K_{24s+3}) = I(24s+3)$.

Case 2. If $r = 2s+1$, then $K_{24s+15} = K_{24s+7} \oplus K_9 \oplus 2(4s+1)K_{4,6}$. By Case 1, and by Lemmas 3.1, 3.5, we have $M(K_{24s+15}) \supseteq M(K_{24s+4}) + M(K_9) + 2(4s+1)M(K_{4,6})$=$\{2x+3|1 \leq x \leq 2(24s^2+13s+1)\} + \{0,4,6,8,10,12\}+2(4s+1)\{0,8\}$=$\{2y+3|0 \leq y \leq 2(24s^2+29s+8)\}$. Therefore $M(K_{24s+15}) = I(24s+15)$. Thus $M(K_{12r+3}) = I(12r+3)$ for each $r \in \mathbb{Z}^+$. □

Lemma 4.4. *$M(K_{12r+4}) = I(12r+4)$, where $r \in \mathbb{Z}^+$.*

Proof. Necessity: The condition $M(K_{12r+4}) \subseteq I(12r+4)$ is trivial, by 1.1. Sufficiency: We have to prove that $M(K_{12r+4}) \supseteq I(12r+4)$. The proof is by induction on r. If $r = 1$, then $M(K_{16}) = I(16)$, by Lemma 3.10. We write $K_{12r+16} = K_{12r+4} \oplus K_{12} \oplus K_{12r+4,12} = K_{12r+4} \oplus K_{12} \oplus (6r+2)K_{4,6}$. From the definition of $I(n)$, we have

$$\begin{aligned}
I(24s+4) &= \left\{\beta \Big| \beta = \frac{(24s+4)(24s+3)}{6} - 2i,\ 0 \leq i \leq \left\lfloor \frac{(24s+4)(24s+3)}{12} - \frac{(24s+4)}{8} \right\rfloor \right\}, \\
&= \{2x+2|3s \leq x \leq 2(24s^2+7s)\}. \\
I(24s+16) &= \left\{\beta \Big| \beta = \frac{(24s+16)(24s+15)}{6} - 2i,\ 0 \leq i \leq \left\lfloor \frac{(24s+16)(24s+15)}{12} - \frac{(24s+16)}{8} \right\rfloor \right\}, \\
&= \left\{2x \Big| \left\lceil \frac{6s+3}{2} \right\rceil \leq x \leq 2(24s^2+31s+10) \right\}. \\
I(24s+28) &= \left\{\beta \Big| \beta = \frac{(24s+28)(24s+27)}{6} - 2i,\ 0 \leq i \leq \left\lfloor \frac{(24s+28)(24s+27)}{12} - \frac{(24s+28)}{8} \right\rfloor \right\}, \\
&= \left\{2x+2 \Big| \left\lceil \frac{6s+5}{2} \right\rceil \leq x \leq 2(24s^2+55s+31) \right\}.
\end{aligned}$$

Case 1. If $r = 2s$, then $K_{24s+16} = K_{24s+4} \oplus K_{12} \oplus (12s+2)K_{4,6}$. By the induction hypothesis, and

Lemmas 3.1, 3.7, we have $M(K_{24s+16}) \supseteq M(K_{24s+4}) + M(K_{12}) + (12s+2)M(K_{4,6})=\{2x+2|3s \leq x \leq 2(24s^2+7s)\} + \{2x|2 \leq x \leq 11\} \cup \{0\} + (12s+2)\{0,8\} = \big\{2y\big|\left\lceil\frac{6s+3}{2}\right\rceil \leq y \leq 2(24s^2+31s+10)\big\}$. Therefore $M(K_{24s+16}) = I(24s+16)$.

Case 2. If $r = 2s+1$, then $K_{24s+28} = K_{24s+16} \oplus K_{12} \oplus (12s+8)K_{4,6}$. By Case 1, and Lemmas 3.1, 3.7, we have $M(K_{24s+28}) \supseteq M(K_{24s+16}) + M(K_{12}) + (12s+8)M(K_{4,6})=\big\{2x\big|\left\lceil\frac{6s+3}{2}\right\rceil \leq x \leq 2(24s^2+31s+10)\big\} + \{2x|2 \leq x \leq 11\} \cup \{0\} + (12s+8)\{0,8\}=\big\{2y+2\big|\left\lceil\frac{6s+5}{2}\right\rceil \leq x \leq 2(24s^2+55s+31)\big\}$. Therefore $M(K_{24s+28}) = I(24s+28)$. Thus $M(K_{12r+4}) = I(12r+4)$ for each $r \in \mathbb{Z}^+$. □

Lemma 4.5. *$M(K_{12r+6}) = I(12r+6)$, where $r \in \mathbb{Z}^+ \cup \{0\}$.*

Proof. Necessity: The condition $M(K_{12r+6}) \subseteq I(12r+6)$ is trivial, by 1.1. Sufficiency: We have to prove that $M(K_{12r+6}) \supseteq I(12r+6)$. The proof is by induction on r. If $r=0$ and $r=1$, then $M(K_k) = I(k)$, where $k = 6, 18$ by Lemmas 3.3 and 3.11. We can write $K_{12r+18} = K_{12r+6} \oplus K_{12} \oplus K_{12r+6,12} = K_{12r+6} \oplus K_{12} \oplus (6r+3)K_{4,6}$. From the definition of $I(n)$, we have

$$\begin{aligned}
I(24s+6) &= \left\{\beta\middle|\beta = \frac{(24s+6)(24s+5)}{6} - 2i,\ 0 \leq i \leq \left\lfloor \frac{(24s+6)(24s+5)}{12} - \frac{(24s+6)}{8}\right\rfloor\right\},\\
&= \left\{2x+1\middle| \left\lceil\frac{6s+1}{2}\right\rceil \leq x \leq 2(24s^2+11s+1)\right\}.\\
I(24s+18) &= \left\{\beta\middle|\beta = \frac{(24s+18)(24s+17)}{6} - 2i,\ 0 \leq i \leq \left\lfloor \frac{(24s+18)(24s+17)}{12} - \frac{(24s+18)}{8}\right\rfloor\right\},\\
&= \left\{2x+3\middle| \left\lceil\frac{6s+1}{2}\right\rceil \leq x \leq 2(24s^2+35s+12)\right\}.\\
I(24s+30) &= \left\{\beta\middle|\beta = \frac{(24s+30)(24s+29)}{6} - 2i,\ 0 \leq i \leq \left\lfloor \frac{(24s+30)(24s+29)}{12} - \frac{(24s+30)}{8}\right\rfloor\right\},\\
&= \left\{2x+1\middle| \left\lceil\frac{6s+7}{2}\right\rceil \leq x \leq 2(24s^2+59s+36)\right\}.
\end{aligned}$$

Case 1. If $r = 2s$, then $K_{24s+18} = K_{24s+6} \oplus K_{12} \oplus (12s+3)K_{4,6}$. By the induction hypothesis, and Lemmas 3.1 3.7, we have $M(K_{24s+18}) \supseteq M(K_{24s+6}) + M(K_{12}) + (12s+3)M(K_{4,6})=\big\{2x+1\big|\left\lceil\frac{6s+1}{2}\right\rceil \leq x \leq 2(24s^2+11s+1)\big\}+\{2x|2 \leq x \leq 11\}\cup\{0\}+(12s+3)\{0,8\}=\big\{2y+3\big|\left\lceil\frac{6s+1}{2}\right\rceil \leq y \leq 2(24s^2+35s+12)\big\}$. Therefore $M(K_{24s+18}) = I(24s+18)$.

Case 2. If $r = 2s+1$, then $K_{24s+30} = K_{24s+18} \oplus K_{12} \oplus (12s+9)K_{4,6}$. By Case 1, and Lemmas 3.1, 3.7, we have $M(K_{24s+30}) \supseteq M(K_{24s+18}) + M(K_{12}) + (12s+9)M(K_{4,6})=\big\{2x+3\big|\left\lceil\frac{6s+1}{2}\right\rceil \leq x \leq 2(24s^2+35s+12)\big\}+\{2x|2 \leq x \leq 11\}\cup\{0\}+(12s+9)\{0,8\}=\big\{2y+1\big|\left\lceil\frac{6s+7}{2}\right\rceil \leq y \leq 2(24s^2+59s+36)\big\}$. Therefore $M(K_{24s+30}) = I(24s+30)$. Thus $M(K_{12r+6}) = I(12r+6)$ for each $r \in \mathbb{Z}^+ \cup \{0\}$. □

Lemma 4.6. *$M(K_{12r+7}) = I(12r+7)$, where $r \in \mathbb{Z}^+ \cup \{0\}$.*

Proof. Necessity: The condition $M(K_{12r+7}) \subseteq I(12r+7)$ is trivial, by 1.1. Sufficiency: We have to prove that $M(K_{12r+7}) \supseteq I(12r+7)$. The proof is by induction on r. If $r=0$, then $M(K_7) = I(7)$, by Lemma 3.4. We can write $K_{12r+19} = K_{12r+7} \oplus K_{13} \oplus K_{12r+6,12} = K_{12r+7} \oplus K_{13} \oplus (6r+3)K_{4,6}$. From the definition of $I(n)$, we have

$$\begin{aligned}
I(24s+7) &= \left\{\beta\middle|\beta = \frac{(24s+7)(24s+6)}{6} - 2i,\ 0 \leq i \leq \left\lfloor \frac{(24s+7)(24s+6)-12}{12}\right\rfloor,\ \beta \neq 1,2\right\},\\
&= \{2x+3|0 \leq x \leq 2(24s^2+13s+1)\}.\\
I(24s+19) &= \left\{\beta\middle|\beta = \frac{(24s+19)(24s+18)}{6} - 2i,\ 0 \leq i \leq \left\lfloor \frac{(24s+19)(24s+18)-12}{12}\right\rfloor,\ \beta \neq 1,2\right\},\\
&= \{2x+3|0 \leq x \leq 2(24s^2+37s+13)+1\}.\\
I(24s+31) &= \left\{\beta\middle|\beta = \frac{(24s+31)(24s+30)}{6} - 2i,\ 0 \leq i \leq \left\lfloor \frac{(24s+31)(24s+30)-12}{12}\right\rfloor,\ \beta \neq 1,2\right\},\\
&= \{2x+3|0 \leq x \leq 2(24s^2+61s+38)\}.
\end{aligned}$$

Case 1. If $r = 2s$, then $K_{24s+19} = K_{24s+7} \oplus K_{13} \oplus (12s+3)K_{4,6}$. By induction hypothesis, and Lemmas 3.1, 3.8, we have $M(K_{24s+19}) \supseteq M(K_{24s+7}) + M(K_{13}) + (12s+3)M(K_{4,6})=\{2x+3|0 \leq x \leq$

$2(24s^2 + 13s + 1)\} + \{2x|2 \le x \le 13\} \cup \{0\} + (12s + 3)\{0, 8\}=\{2y + 3|0 \le y \le 2(24s^2 + 37s + 13) + 1\}$. Therefore $M(K_{24s+19}) = I(24s + 19)$.

Case 2. If $r = 2s + 1$, then $K_{24s+31} = K_{24s+19} \oplus K_{13} \oplus (12s + 9)K_{4,6}$. By Case 1, and Lemmas 3.1, 3.8, we have $M(K_{24s+31}) \supseteq M(K_{24s+19}) + M(K_{13}) + (12s + 9)M(K_{4,6})=\{2x + 3|0 \le x \le 2(24s^2 + 37s + 13) + 1\} + \{2x|2 \le x \le 13\} \cup \{0\} + (12s + 9)\{0, 8\}=\{2y + 3|0 \le y \le 2(24s^2 + 61s + 38)\}$. Therefore $M(K_{24s+31}) = I(24s + 31)$. Thus $M(K_{12r+7}) = I(12r + 7)$ for each $r \in \mathbb{Z}^+ \cup \{0\}$. □

Lemma 4.7. *$M(K_{12r+9}) = I(12r + 9)$, where $r \in \mathbb{Z}^+ \cup \{0\}$.*

Proof. Necessity: The condition $M(K_{12r+9}) \subseteq I(12r + 9)$ is trivial, by 1.1. Sufficiency: We have to prove that $M(K_{12r+9}) \supseteq I(12r + 9)$. The proof is by induction on r. If $r = 0$, then $M(K_9) = I(9)$, by Lemma 3.5. We can write $K_{12r+21} = K_{12r+9} \oplus K_{13} \oplus K_{12r+8,12} = K_{12r+9} \oplus K_{13} \oplus (6r + 4)K_{4,6}$. From the definition of $I(n)$, we have

$$\begin{aligned}
I(24s+9) &= \left\{\beta\middle|\beta = \frac{(24s+9)(24s+8)}{6} - 2i,\ 0 \le i \le \left\lfloor \frac{(24s+9)(24s+8)}{12}\right\rfloor,\ \beta \ne 1,2\right\},\\
&= \{2x|0 \le x \le 2(24s^2+17s+3), x \ne 1\}.\\
I(24s+21) &= \left\{\beta\middle|\beta = \frac{(24s+21)(24s+20)}{6} - 2i,\ 0 \le i \le \left\lfloor \frac{(24s+21)(24s+20)}{12}\right\rfloor,\ \beta \ne 1,2\right\},\\
&= \{2x|0 \le x \le 2(24s^2+41s+17)+1, x \ne 1\}.\\
I(24s+33) &= \left\{\beta\middle|\beta = \frac{(24s+33)(24s+32)}{6} - 2i,\ 0 \le i \le \left\lfloor \frac{(24s+33)(24s+32)}{12}\right\rfloor,\ \beta \ne 1,2\right\},\\
&= \{2x|0 \le x \le 2(24s^2+65s+44), x \ne 1\}.
\end{aligned}$$

Case 1. If $r = 2s$, then $K_{24s+21} = K_{24s+9} \oplus K_{13} \oplus (12s + 4)K_{4,6}$. By the induction hypothesis, and Lemmas 3.1, 3.8, we have $M(K_{24s+21}) \supseteq M(K_{24s+9}) + M(K_{13}) + (12s + 4)M(K_{4,6})=\{2x|0 \le x \le 2(24s^2+17s+3), x \ne 1\}+\{2x|2 \le x \le 13\}\cup\{0\}+(12s+4)\{0, 8\}=\{2y|0 \le y \le 2(24s^2+41s+17)+1, y \ne 1\}$. Therefore $M(K_{24s+21}) = I(24s + 21)$.

Case 2. If $r = 2s+1$, then $K_{24s+33} = K_{24s+21}\oplus K_{13}\oplus(12s+10)K_{4,6}$. By Case 1, and Lemmas 3.1, 3.8, we have $M(K_{24s+33}) \supseteq M(K_{24s+21})+M(K_{13})+(12s+10)M(K_{4,6})=\{2x|0 \le x \le 2(24s^2+41s+17)+1, x \ne 1\} + \{2x|2 \le x \le 13\} \cup \{0\} + (12s + 10)\{0, 8\}=\{2y|0 \le y \le 2(24s^2 + 65s + 44), y \ne 1\}$. Therefore $M(K_{24s+33}) = I(24s + 33)$. Thus $M(K_{12r+9}) = I(12r + 9)$ for each $r \in \mathbb{Z}^+ \cup \{0\}$. □

Lemma 4.8. *$M(K_{12r+10}) = I(12r + 10)$, where $r \in \mathbb{Z}^+ \cup \{0\}$.*

Proof. Necessity: The condition $M(K_{12r+10}) \subseteq I(12r + 10)$ is trivial, by 1.1. Sufficiency: We have to prove that $M(K_{12r+10}) \supseteq I(12r + 10)$. The proof is by induction on r. If $r = 0$, then $M(K_{10}) = I(10)$, by Lemma 3.6. We write $K_{12r+10} = K_{12r-8} \oplus K_{18} \oplus K_{12r-8,18} = K_{12r-8} \oplus K_{18} \oplus (9r - 6)K_{4,6}$. From the definition of $I(n)$, we have

$$\begin{aligned}
I(24s-8) &= \left\{\beta\middle|\beta = \frac{(24s-8)(24s-9)}{6} - 2i,\ 0 \le i \le \left\lfloor \frac{(24s-8)(24s-9)}{12} - \frac{(24s-8)}{8}\right\rfloor\right\},\\
&= \left\{2x\middle| \left\lceil \frac{6s-3}{2}\right\rceil \le x \le 2(24s^2-17s+3)\right\}.\\
I(24s+10) &= \left\{\beta\middle|\beta = \frac{(24s+10)(24s+9)}{6} - 2i,\ 0 \le i \le \left\lfloor \frac{(24s+10)(24s+9)}{12} - \frac{(24s+10)}{8}\right\rfloor\right\},\\
&= \left\{2x+3|3s \le x \le 2(24s^2+19s+3)\right\}.\\
I(24s+22) &= \left\{2x+3|3s+2 \le x \le 2(24s^2+43s+18)+1\right\}.
\end{aligned}$$

Case 1. If $r = 2s$, then $K_{24s+10} = K_{24s-8} \oplus K_{18} \oplus (18s - 6)K_{4,6}$. By the induction hypothesis, and Lemmas 3.1 and 3.11, we have $M(K_{24s+10}) \supseteq M(K_{24s-8}) + M(K_{18}) + (18s-6)M(K_{4,6})=\left\{2x\middle| \left\lceil \frac{6s+3}{2}\right\rceil \le x \le 2(24s^2 - 17s + 3)\right\} + \{2x + 1|2 \le x \le 25\} + (18s - 6)\{0, 8\}=\left\{2y + 3|3s \le y \le 2(24s^2 + 19s + 3)\right\}$. Therefore $M(K_{24s+10}) = I(24s + 10)$.

Case 2. If $r = 2s+1$, then $K_{24s+22} = K_{24s+4}\oplus K_{18}\oplus(18s+3)K_{4,6}$. By Lemmas 3.1, 3.11 and 4.4, we have

$M(K_{24s+22}) \supseteq M(K_{24s+4}) + M(K_{18}) + (18s+3)M(K_{4,6})=\{2x+2|3s \leq x \leq 2(24s^2+7s)\} + \{2x+1|2 \leq x \leq 25\} + (18s+3)\{0,8\}=\{2y+3|3s \leq y \leq 2(24s^2+19s+3)\}$. Therefore $M(K_{24s+22}) = I(24s+22)$. Thus $M(K_{12r+10}) = I(12r+10)$, for all integer $n \geq 6$. □

The consequences of 4.1 to 4.8 implies our main result as follows.

Theorem 4.9. *Let $\alpha, \beta \in \mathbb{Z}^+ \cup \{0\}$, and $6 \leq n \in \mathbb{Z}^+$. K_n has a $\{C_6, S_3\}^{\{\alpha,\beta\}}$-decomposition if and only if $6\alpha + 3\beta = \binom{n}{2}$, $\beta \neq 1, 2$ when n is odd, and $\beta \geq \lceil \frac{n}{4} \rceil$ when n is even. That is, $M(K_n) = I(n)$, where $6 \leq n \in \mathbb{Z}^+$.*

Proof. Necessity follows from Theorem 1.1 and the sufficiency follows from our results in Lemmas 4.1 to 4.8. □

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