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On weakly S-2-absorbing filters in lattices

Research Article

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Abstract: Let $\pounds$ be a bounded distributive lattice and S a join closed subset of $\pounds$. Following the concept of weakly S-2-absorbing submodules, we define weakly S-2-absorbing filters of $\pounds$. We will make an intensive investigate the basic properties and possible structures of these filters.

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1. Introduction

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are bounded. Our objective in this paper is to extend the notion of weakly S-2-absorbing property in modules theory to weakly S-2-absorbing property in lattice theory. Indeed, we are interested in investigating weakly S-2-absorbing filters to use other notions of weakly S-2-absorbing and associate which exist in the literature as laid forth in [16].

Prime ideals and submodules have a significant place in ring and module theory. Several generalizations of these concepts have been investigated extensively by many authors see, for example, [1, 2, 4, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16]. In 2003, Anderson and Smith in [1] defined weakly prime ideals which is a generalization of prime ideals (also see [7, 9]). A proper ideal P of a ring R is said to be a weakly prime if $0 \neq xy \in P$ for each $x, y \in R$ implies either $x \in P$ or $y \in P$. Badawi generalized the concept of prime ideals in [4]. We recall from [4] that a proper ideal I of a commutative ring R is said to be a 2-absorbing ideal if whenever $abc \in I$ for $a, b, c \in R$, then $ab \in I$ or $ac \in I$ or $bc \in I$ (also see [6]). In 2019, Hamed and Malek [12] introduced the notion of an S-prime ideal, i.e. let $S \subseteq R$ be a multiplicative set and I an ideal of R disjoint from S. We say that I is S-prime if there exists an $s \in S$ such that for all $a, b \in R$ with $ab \in I$, we have $sa \in I$ or $sb \in I$ (also see [14]). Almahdi et. al. [2] introduced the notion of a weakly S-prime ideal as follows: We say that I is a weakly S-prime ideal of R if there is an element $s \in S$ such that for all $x, y \in R$ if $0 \neq xy \in I$, then $xs \in I$ or $ys \in I$.

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In 2020, Ulucak, Tekir and Koc [15] introduced the notion of an S-2-absorbing submodules, i.e. let $S \subseteq R$ be a multiplicative set and P a submodule of an R-module M with $S \cap (P :_R M) = \emptyset$. We say that P is an S-2-absorbing submodule if there exists an element $s \in S$ and whenever $abm \in P$ for some $a, b \in R$ and $m \in M$, then $sab \in (P :_R M)$ or $sam \in P$ or $sbm \in P$. In 2021, Naji [13] introduced the notion of an S-2-absorbing primary submodules, i.e. let $S \subseteq R$ be a multiplicative set and P a submodule of an R-module M with $S \cap (P :_R M) = \emptyset$. We say that P is an S-2-absorbing primary submodule if there exists an element $s \in S$ and whenever $abm \in P$ for some $a, b \in R$ and $m \in M$, then $sab \in \sqrt{(P :_R M)}$ or $sam \in P$ or $sbm \in P$. In 2023, Sudharshana [16] introduced the notion of a weakly S-2-absorbing submodules, i.e. let $S \subseteq R$ be a multiplicative set and P a submodule of an R-module M with $S \cap (P :_R M) = \emptyset$. We say that P is a weakly S-2-absorbing submodule if there exists an element $s \in S$ and whenever $0 \neq abm \in P$ for some $a, b \in R$ and $m \in M$, then $sab \in (P :_R M)$ or $sam \in P$ or $sbm \in P$.

Let $\pounds$ be a bounded distributive lattice. We say that a subset $S \subseteq \pounds$ is *join closed* if $0 \in S$ and $s_1 \vee s_2 \in S$ for all $s_1, s_2 \in S$ (if P is a prime filter of $\pounds$, then $\pounds \setminus P$ is a join closed subset of $\pounds$). Among many results in this paper, the first, preliminaries section contains elementary observations needed later on. Section 3 is dedicated to the investigation of the some basic properties of weakly S-2-absorbing filters. At first, we give the definition of weakly S-2-absorbing filters (Definition 3.1) and provide an example (Example 3.4) of a weakly S-2-absorbing filter of $\pounds$ that is not a weakly 2-absorbing filter. It is shown (Theorem 3.7) that if S is a join closed subset of $\pounds$, then the intersection of two weakly S-prime filter is a weakly S-2-absorbing filter. We provides some condition under which a weakly S-2-absorbing filter is S-2-absorbing (see Theorem 3.9). Also, in Theorem 3.12, we give two other characterizations of weakly S-2-absorbing filters. The rest of this section, we investigate a more explicit description of the weakly S-2-absorbing filters of $\pounds$ (see Lemma 3.14, Lemma 3.15, Proposition 3.16 and Theorem 3.17). We continue in Section 4 with the investigation of the stability of weakly S-2-absorbing filters in various lattice-theoretic constructions. Indeed, we investigate the behavior of weakly S-2-absorbing filters under homomorphism, in factor lattices and in cartesian products of lattices (see Theorem 4.3, Theorem 4.5 and Theorem 4.6).

2. Preliminaries

A poset $(\pounds, \leq)$ is a lattice if $\sup\{a, b\} = a \vee b$ and $\inf\{a, b\} = a \wedge b$ exist for all $a, b \in \pounds$ (and call $\wedge$ the meet and $\vee$ the join). A lattice $\pounds$ is *complete* when each of its subsets X has a join and a meet in $\pounds$. Setting $X = \pounds$, we see that any non-void complete lattice contains a least element 0 and greatest element 1 (in this case, we say that $\pounds$ is a lattice with 0 and 1). A lattice $\pounds$ is called a *distributive* lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in $\pounds$ (equivalently, $\pounds$ is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in $\pounds$). A non-empty subset F of a lattice $\pounds$ is called a *filter*, if for $a \in F$, $b \in \pounds$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if $\pounds$ is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of $\pounds$). A proper filter F of $\pounds$ is called *prime* if $x \vee y \in F$, then $x \in F$ or $y \in F$. A proper filter F of $\pounds$ is said to be *maximal* if G is a filter in $\pounds$ with $F \subsetneqq G$, then $G = \pounds$. The intersection of all filters containing a given subset A of $\pounds$ is the *filter generated by* it, is denoted by $T(A)$. A filter F is called *finitely generated* if there is a finite subset A of F such that $F = T(A)$. A proper filter F of a lattice $\pounds$ is called a 2-*absorbing* (resp. *weakly* 2-*absorbing*) filter if whenever $a, b, c \in \pounds$ and $a \vee b \vee c \in F$ (resp. $1 \neq a \vee b \vee c \in F$), then $a \vee b \in F$ or $a \vee c \in F$ or $b \vee c \in F$. Let P be a filter of $\pounds$ and S a join closed subset of $\pounds$ disjoint with S. We say that P is an S-*prime* (resp. weakly S-prime) filter of $\pounds$ if there is an element $s \in S$ such that for all $x, y \in \pounds$ if $x \vee y \in P$ (resp. $1 \neq x \vee y \in P$), then $x \vee s \in P$ or $y \vee s \in P$. A filter P is said to be S-2-absorbing if $P \cap S = \emptyset$ and there exists a fixed $s \in S$ such that for any $x, y, z \in \pounds$ with $x \vee y \vee z \in P$, then $s \vee x \vee y \in P$ or $s \vee x \vee z \in P$ or $s \vee y \vee z \in P$.

A lattice $\pounds$ with 1 is called $\pounds$-*domain* if $a \vee b = 1$ $(a, b \in \pounds)$, then $a = 1$ or $b = 1$ (so $\pounds$ is $\pounds$-domain if and only if $\{1\}$ is a prime filter of $\pounds$). If $x \in \pounds$, then a *complement* of x in $\pounds$ is an element $y \in \pounds$ such that $x \vee y = 1$ and $x \wedge y = 0$. The lattice $\pounds$ is *complemented* if every element of $\pounds$ has a complement in $\pounds$. If $\pounds$ and $\pounds'$ are lattices, then a *lattice homomorphism* $f : \pounds \to \pounds'$ is a map from $\pounds$ to $\pounds'$ satisfying $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$ for $x, y \in \pounds$. First we need the following lemmas proved in [3, 5, 6, 8, 10].

Lemma 2.1. *Let $£$ be a lattice.*

(1) A non-empty subset F of $£$ is a filter of $£$ if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in £$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in £$.

(2) Let A be an arbitrary non-empty subset of $£$. Then

$$T(A) = \{x \in £ : \ a_1 \wedge a_2 \wedge \cdots \wedge a_n \leq x \text{ for some } a_i \in A \ (1 \leq i \leq n)\}.$$

Moreover, if F is a filter and A is a subset of $£$ with $A \subseteq F$, then $T(A) \subseteq F$, $T(F) = F$ and $T(T(A)) = T(A)$

(3) If $\{F_i\}_{i \in \Delta}$ is a chain of filters of $£$, then $\bigcup_{i \in \Delta} F_i$ is a filter of $£$.

Lemma 2.2. *Let F, G be filters of $£$ and $x \in £$. The following hold:*

(1) $F \vee G = \{a \vee b : a \in F, b \in G\}$ and $x \vee F = \{a \vee y : y \in F\}$ are filters of $£$ with $F \vee G = F \cap G$.

(2) If $£$ is distributive, then $F \wedge G = \{a \wedge b : a \in F, b \in G\}$ is a filter of $£$ with $F, G \subseteq F \wedge G$

(3) If $£$ is distributive, F, G are filters of $£$ and $y \in £$, then $(G :_£ F) = \{x \in £ : x \vee F \subseteq G\}$ and $(F :_£ T(\{y\})) = (F :_£ y) = \{a \in £ : a \vee y \in F\}$ are filters of $£$.

(4) If $£$ is distributive, G, F_1, F_2 are filters of $£$, then $G \vee (F_1 \wedge F_2) = (G \vee F_1) \wedge (G \vee F_2)$.

Lemma 2.3. *[11, Lemma 3.13] Let $£_1$ and $£_2$ be lattices and $f : £_1 \to £_2$ be a lattice homomorphism such that $f(1) = 1$. The following hold:*

(1) $\mathrm{Ker}(f) = \{x \in £_1 : f(x) = 1\}$ is a filter of $£_1$;

(2) If f is injective, then $\mathrm{Ker}(f) = \{1\}$;

(3) If $£_1$ is a complemented lattice, then f is injective if and only if $\mathrm{Ker}(f) = \{1\}$.

Assume that $(£_1, \leq_1), (£_2, \leq_2), \cdots, (£_n, \leq_n)$ are lattices ($n \geq 2$) and let $£ = £_1 \times £_2 \times \cdots \times £_n$. We set up a partial order $\leq_c$ on $£$ as follows: for each $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in £$, we write $x \leq_c y$ if and only if $x_i \leq_i y_i$ for each $i \in \{1, 2, \cdots, n\}$. The following notation below will be kept in this paper: It is straightforward to check that $(£, \leq_c)$ is a lattice with $x \vee_c y = (x_1 \vee y_1, x_2 \vee y_2, \cdots, x_n \vee y_n)$ and $x \wedge_c y = (x_1 \wedge y_1, \cdots, x_n \wedge y_n)$. In this case, we say that $£$ is a *decomposable lattice.*

Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. If F is a filter of a lattice $(£, \leq)$, we define a relation on $£$, given by $x \sim y$ if and only if there exist $a, b \in F$ satisfying $x \wedge a = y \wedge b$. Then $\sim$ is an equivalence relation on $£$, and we denote the equivalence class of a by $a \wedge F$ and these collection of all equivalence classes by $£/F$. We set up a partial order $\leq_Q$ on $£/F$ as follows: for each $a \wedge F, b \wedge F \in £/F$, we write $a \wedge F \leq_Q b \wedge F$ if and only if $a \leq b$. The following notation below will be kept in this paper: It is straightforward to check that $(£/F, \leq_Q)$ is a lattice with $(a \wedge F) \vee_Q (b \wedge F) = (a \vee b) \wedge F$ and $(a \wedge F) \wedge_Q (b \wedge F) = (a \wedge b) \wedge F$ for all elements $a \wedge F, b \wedge F \in £/F$. Note that $e \wedge F = F = 1 \wedge F$ if and only if $e \in F$ (see [8, Remark 4.2 and Lemma 4.3]).

3. Characterization of weakly S-2-absorbing filters

In this section, we collect some basic properties concerning weakly S-2-absorbing filters. We introduce the reader the following definition.

Definition 3.1. *Let P be a filter of $£$ and S a join closed subset of $£$. A filter P is said to be weakly S-2-absorbing if $P \cap S = \emptyset$ and there exists a fixed $s \in S$ such that for any $x, y, z \in £$ with $1 \neq x \vee y \vee z \in P$, then $s \vee x \vee y \in P$ or $s \vee x \vee z \in P$ or $s \vee y \vee z \in P$.*

Example 3.2. *(1) A weakly* 2*-absorbing filter* P *of* $\pounds$ *is weakly* S*-2-absorbing for each join closed subset* S *of* $\pounds$ *such that* $S \cap P = \emptyset$*.*

(2) If $S = \{0\}$*, then the weakly* 2*-absorbing and the weakly* S*-2-absorbing filters of* $\pounds$ *are the same.*

(3) It is clear that every S*-2-absorbing filter is a weakly* S*-2-absorbing filter. Since the filter* $\{1\}$ *is (by definition) a weakly* S*-2-absorbing filter of any lattice, hence the converse is not true in general.*

(4) Every weakly S*-prime filter is a weakly* S*-2-absorbing filter. Indeed, let* P *be an* S*-prime filter of* $\pounds$ *and* $a, b, c \in \pounds$ *such that* $1 \neq a \vee b \vee c \in P$*. Then there exists* $s \in S$ *such that* $s \vee a \in P$ *(so* $s \vee a \vee b, s \vee a \vee c \in P$*, as* P *is a filter) or* $s \vee b \vee c \in P$*, as needed.*

Example 3.3. *Let* $\pounds_1 = \{0, a, b, c, d, 1\}$ *be a lattice with the relations* $0 \leq a \leq d \leq 1$*,* $0 \leq b \leq d \leq 1$*,* $0 \leq c \leq 1$ *and* $a \wedge b = a \wedge c = d \wedge c = c \wedge b = 0$*. Suppose that* $\pounds = \pounds_1 \times \pounds_1$*,* $P = \{b, d, 1\} \times \{1\}$ *and* $S = \{0, c\} \times \{0, c\}$*; so* P *is a filter of* $\pounds$ *with* $P \cap S = \emptyset$*. Then* P *is an weakly* S*-2-absorbing filter. Indeed, let* $(1,1) \neq (a_1, b_1) \vee_c (a_2, b_2) \vee_c (a_3, b_3) \in P$ *for some* $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in \pounds$*. Then* $(c, c) \in S$*,* $b_1 \vee b_2 \vee b_3 = 1$ *and* $a_1 \vee a_2 \vee a_3 \in \{b, d\}$*. Now consider the following two cases.*

Case 1: *There exists* $i \in \{1, 2, 3\}$ *such that* $b_i = 1$*, say* $b_1 = 1$*. If* $a_1 \vee a_2 \vee a_3 = b$ *and there exists* $i \in \{1, 2, 3\}$ *such that* $a_i = 0$*, say* $a_1 = 0$*, then either* $(c, c) \vee_c (a_1, b_1) \vee_c (a_2, b_2) = (1, 1) \in P$ *or* $(c, c) \vee_c (a_1, b_1) \vee_c (a_3, b_3) = (1, 1) \in P$*. So we may assume that* a_1, a_2, a_3 *are non-zero elements. Then* $(c, c) \vee_c (a_1, b_1) \vee_c (a_2, b_2) = (1, 1) \in P$*. Similarly, for* $a_1 \vee_2 \vee a_3 = d$*.*

Case 2: $b_1 \neq 1, b_2 \neq 1$ *and* $b_3 \neq 1$*. Then there exists* $i \in \{1, 2, 3\}$ *such that* $b_i = c$*, say* $b_1 = c$ *and either* $b_2 \in \{a, b, d\}$ *or* $b_3 \in \{a, b, d\}$*. Let* $b_1 = c$ *and* $b_2 \in \{a, b, d\}$*. If* $a_1 \vee a_2 \vee a_3 = b$ *and* $a_1 = 0$*, then* $(c, c) \vee_c (a_1, b_1) \vee_c (a_2, b_2) = (1, 1) \in P$ *or* $(c, c) \vee_c (a_1, b_1) \vee_c (a_3, b_3) = (1, 1) \in P$*. If* $a_1 \neq 0$*, then* $(c, c) \vee_c (a_1, b_1) \vee_c (a_2, b_2) = (1, 1) \in P$*. Similarly, for* $b_1 = c$ *and* $b_3 \in \{a, b, d\}$*. By an argument like that as above, when* $a_1 \vee a_2 \vee a_3 = d$ *the result holds.*

Example 3.4. *Let* $\pounds_1 = \{0, a, b, c, 1\}$ *be a lattice with the relations* $0 \leq a \leq c \leq 1$*,* $0 \leq b \leq c \leq 1$*,* $a \vee b = c$ *and* $a \wedge b = 0$*. Suppose that* $\pounds = \pounds_1 \times \pounds_1$*,* $P = \{1, c\} \times \{1\}$ *and* $S = \{0, a\} \times \{0, a\}$*; so* P *is a filter of* $\pounds$ *with* $P \cap S = \emptyset$*. Then* P *is a weakly* S*-2-absorbing filter. Indeed, let* $(1,1) \neq (a_1, b_1) \vee_c (a_2, b_2) \vee_c (a_3, b_3) = (c, 1) \in P$ *for some* $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in \pounds$*. Then* $(a, a) \in S$*,* $b_1 \vee b_2 \vee b_3 = 1$ *and* $a_1 \vee a_2 \vee a_3 = c$*. Since* $b_1 \vee_2 \vee b_3 = 1$*, we conclude that there exists* $i \in \{1, 2, 3\}$ *such that* $b_i = 1$*, say* $b_1 = 1$*. Then either* $(a, a) \vee_c (a_1, b_1) \vee_c (a_2, b_2) \in P$ *or* $(a, a) \vee_c (a_1, b_1) \vee_c (a_3, b_3) \in P$*, as needed.*

On the other hand, P *is not a weakly* 2*-absorbing filter since* $(1, 1) \neq (a, b) \vee (b, 0) \vee (0, 1) = (c, 1) \in P$ *but neither* $(a, b) \vee (b, 0) = (c, b) \in P$ *nor* $(a, b) \vee (0, 1) = (a, 1) \in P$ *nor* $(b, 0) \vee (0, 1) = (b, 1) \in P$*. Thus a weakly* S*-2-absorbing filter need not be a weakly* 2*-absorbing filter.*

Example 3.5. *Let* $S' \subseteq S$ *be join closed subsets of* $\pounds$ *and* P *a filter of* $\pounds$ *disjoint with* S*. It s clear that if* P *is a weakly* S'*-2-absorbing filter of* $\pounds$*, then* P *is a weakly* S*-2-absorbing filter. However, the converse is not true in general. Indeed, suppose that* $\pounds$ *is the lattice as in Example 3.4 and let* $S' = \{(0, 0)\} \subseteq S = \{0, a\} \times \{0, a\}$*. Then* $P = \{1, c\} \times \{1\}$ *is a weakly* S*-2-absorbing filter of* $\pounds$ *but not a weakly* S'*-2-absorbing filter of* $\pounds$*.*

Proposition 3.6. *Let* $S' \subseteq S$ *be join closed subsets of* $\pounds$ *such that for any* $s \in S$*, there exists* $t \in S$ *satisfying* $s \vee t \in S'$*. If* P *is a weakly* S*-2-absorbing filter of* $\pounds$*, then* P *is a weakly* S'*-2-absorbing filter of* $\pounds$*.*

Proof. Let $x, y, z \in \pounds$ such that $1 \neq x \vee y \vee z \in P$. Then there exists $s \in S$ such that $s \vee x \vee y \in P$ or $s \vee x \vee z \in P$ or $s \vee y \vee z \in P$. By the hypothesis, there exists a $t \in S$ such that $s \vee t \in S'$ and then $s \vee t \vee x \vee y \in P$ or $s \vee t \vee x \vee z \in P$ or $s \vee t \vee y \vee z \in P$, as P is a filter. This shows that P is a weakly S'-2-absorbing filter. □

Theorem 3.7. *If* S *is a join closed subset of* $\pounds$*, then the intersection of two weakly* S*-prime filter is a weakly* S*-2-absorbing filter.*

Proof. Let P_1, P_2 be two weakly S-prime filters of $\pounds$ and $P = P_1 \cap P_2$. Let $a, b, c \in \pounds$ such that $1 \neq a \vee b \vee c \in P$. Since P_1 is weakly S-prime and $1 \neq a \vee b \vee c \in P_1$, there exists $s_1 \in S$ such that

$s_1 \vee a \in P_1$ or $s_1 \vee b \vee c \in P_1$. Again as P_2 is weakly S-prime and $1 \neq a \vee b \vee c \in P_2$ there exists $s_2 \in S$ such that $s_2 \vee b \in P_2$ or $s_2 \vee a \vee c \in P_2$. We split the proof into four cases.

Case 1: $s_1 \vee a \in P_1$, $s_1 \vee b \vee c \notin P_1$, $s_2 \vee b \in P_2$ and $s_2 \vee a \vee c \notin P_2$. Now, put $s = s_1 \vee s_2 \in S$. Then $s \vee a \vee b \in P_1 \cap P_2 = P$, as P_1 and P_2 are filters.

Case 2: $s_1 \vee a \in P_1$, $s_1 \vee b \vee c \notin P_1$, $s_2 \vee b \notin P_2$ and $s_2 \vee a \vee c \in P_2$. Set $s = s_1 \vee s_2$. Then $s \vee a \vee c \in P$.

Case 3: $s_1 \vee a \notin P_1$, $s_1 \vee b \vee c \in P_1$, $s_2 \vee b \in P_2$ and $s_2 \vee a \vee c \notin P_2$. It is easy to see that $s \vee b \vee c \in P$, where $s = s_1 \vee s_2$.

Case 4: $s_1 \vee a \notin P_1$, $s_1 \vee b \vee c \in P_1$, $s_2 \vee b \notin P_2$ and $s_2 \vee a \vee c \in P_2$. If $s_1 \vee a \vee c \in P_1$, then $s \vee a \vee c \in P$ and so we are done, where $s = s_1 \vee s_2$. So we may assume that $s_1 \vee a \vee c \notin P_1$. Since P_1 is weakly S-prime, $1 \neq a \vee b \vee c \in P_1$ and $s_1 \vee a \vee c \notin P_1$, we conclude that $s_1 \vee b \in P_1$. Similarly, $s_2 \vee a \in P_2$. Therefore, $s \vee a \vee b \in P$, where $s = s_1 \vee s_2$. Thus P is weakly S-2-absorbing. □

An element x of $\pounds$ is called identity join of a lattice $\pounds$, if there exists $1 \neq y \in \pounds$ such that $x \vee y = 1$. The set of all identity joins of a lattice $\pounds$ is denoted by $\mathrm{I}(\pounds)$.

Proposition 3.8. *Let P be a filter of $\pounds$, S a join closed subset of $\pounds$ disjoint with P and $S \cap \mathrm{I}(\pounds) = \emptyset$. The following assertions are equivalent:*

(1) P is a weakly S-2-absorbing filter of $\pounds$;

(2) $(P :_{\pounds} s)$ is a weakly 2-absorbing filter of $\pounds$ for some $s \in S$.

Proof. (1) $\Rightarrow$ (2) Let P be a weakly S-2-absorbing filter of $\pounds$. Then we keep in mind that there exists a fixed $s \in S$ that satisfies the weakly S-2-absorbing condition. Now, we show that $(P :_{\pounds} s)$ is a weakly 2-absorbing filter of $\pounds$. Let $x, y, z \in \pounds$ such that $1 \neq x \vee y \vee z \in (P :_{\pounds} s)$ (so $x \vee y \vee z \vee s \neq 1$, as $S \cap \mathrm{I}(\pounds) = \emptyset$). Then $1 \neq x \vee y \vee z \vee s \in P$ gives $s \vee x \vee y \in P$ or $s \vee x \vee (s \vee z) = s \vee x \vee z \in P$ or $s \vee y \vee (s \vee z) = s \vee y \vee z \in P$ which means that $x \vee y \in (p :_{\pounds} s)$ or $x \vee z \in (P :_{\pounds} s)$ or $y \vee z \in (P :_{\pounds} s)$. Thus $(P :_{\pounds} s)$ is a weakly 2-absorbing filter of $\pounds$. The implication of (2) $\Rightarrow$ (1) is clear. □

The following theorem provides some condition under which a weakly S-2-absorbing filter is S-2-absorbing.

Theorem 3.9. *Let S be a join closed subset of $\pounds$ and P be a weakly S-2-absorbing filter of $\pounds$. If P is not S-2-absorbing, then $P = \{1\}$. In particular, the only weakly S-2-absorbing filters of $\pounds$ that are not S-2-absorbing can only be $\{1\}$.*

Proof. Let P be a weakly S-2-absorbing filter of $\pounds$ and assume that $s \in S$ satisfies weakly S-2-absorbing condition. On the contrary, assume that $P \neq \{1\}$. It suffices to show that P is S-2-absorbing. Let $a, b, c \in \pounds$ such that $a \vee b \vee c \in P$. If $1 \neq a \vee b \vee c \in P$, then P is weakly S-2-absorbing gives $s \vee a \vee b \in P$ or $s \vee a \vee c \in P$ or $s \vee b \vee c \in P$. Now, suppose that $a \vee b \vee c = 1$. Since $P \neq \{1\}$, there exists $p \in P$ such that $p \neq 1$. Then $1 \neq (a \wedge p) \vee (b \wedge p) \vee (c \wedge p) = p \in P$ gives $s \vee (a \wedge p) \vee (b \wedge p) = (s \vee a \vee b) \wedge (s \vee p) \in P$ or $(s \vee a \vee c) \wedge (s \vee p) \in P$ or $(s \vee b \vee c) \wedge (s \vee p) \in P$. Therefore, $s \vee a \vee b \in P$ or $s \vee a \vee c \in P$ or $s \vee b \vee c \in P$ by lemma 2.1 (1). This shows that P is an S-2-absorbing filter, as required. □

Corollary 3.10. *([6], Theorem 2.2) If P is a weakly 2-absorbing filter that is not 2-absorbing, then $P = \{1\}$.*

Proof. Take $S = \{0\}$ in Theorem 3.9. □

Theorem 3.11. *Let S be a join closed subset of $\pounds$ and P be a weakly S-2-absorbing filter of $\pounds$. If $a, b, c \in \pounds$ with $a \vee b \vee c = 1$ and $s \vee a \vee b, s \vee a \vee c, s \vee b \vee c \notin P$ for any $s \in S$, then the following hold:*

(1) $(a \vee b) \vee P = (a \vee c) \vee P = (b \vee c) \vee P = \{1\}$;

(2) $a \vee P = b \vee P = c \vee P = \{1\}$.

Proof. (1) On the contrary, assume that $(a \vee b) \vee P \neq \{1\}$. Then $a \vee b \vee p \neq 1$ for some $p \in P$. Since $1 \neq a \vee b \vee p = (a \vee b) \vee (p \wedge c) \in P$, there exists $s \in S$ such that $(s \vee a) \vee (c \wedge p) = (s \vee a \vee c) \wedge (s \vee a \vee p) \in P$ or $(s \vee b) \vee (c \wedge p) = (s \vee b \vee c) \wedge (s \vee b \vee p) \in P$ or $s \vee a \vee b \in P$ which implies that $s \vee a \vee c \in P$ or $s \vee b \vee c \in P$ or $s \vee a \vee b \in P$ by Lemma 2.1 (1) which is impossible. Thus $(a \vee b) \vee P = \{1\}$. Similarly, $(a \vee c) \vee P = (b \vee c) \vee P = \{1\}$.

(2) If $a \vee P \neq \{1\}$, then $a \vee p \neq 1$ for some $p \in P$. Since $1 \neq a \vee p = a \vee (b \wedge p) \vee (c \wedge p) \in P$, we conclude that there exists $s \in S$ such that $(s \vee a) \vee (b \wedge p) = (s \vee a \vee b) \wedge (s \vee a \vee p) \in P$ or $(s \vee a) \vee (c \wedge p) = (s \vee a \vee c) \wedge (s \vee a \vee p) \in P$ or $s \vee (b \wedge p) \vee (c \wedge p) = (s \vee b \vee c) \wedge (s \vee p) \in P$ which gives $s \vee a \vee c \in P$ or $s \vee b \vee c \in P$ or $s \vee a \vee b \in P$ by Lemma 2.1 (1) which is a contradiction. Hence $a \vee P = \{1\}$. Similarly, $b \vee P = c \vee P = \{1\}$. □

We next give two other characterizations of weakly S-2-absorbing filters.

Theorem 3.12. *Let P be a filter of $\pounds$ and S a join closed subset of $\pounds$ disjoint with P. The following assertions are equivalent:*

(1) P is a weakly S-2-absorbing filter of $\pounds$;

(2) For any $a, b \in \pounds$, there exists $s \in S$ such that if $s \vee a \vee b \notin P$, then $(P :_\pounds a \vee b) = (1 :_\pounds a \vee b)$ or $(P :_\pounds a \vee b) \subseteq (P :_\pounds s \vee a)$ or $(P :_\pounds a \vee b) \subseteq (P :_\pounds s \vee b)$;

(3) For any $a, b \in \pounds$ and for any filter F of $\pounds$, there exists $s \in S$ such that, if $\{1\} \neq (a \vee b) \vee F \subseteq P$, then $s \vee a \vee b \in P$ or $F \subseteq (P :_\pounds s \vee a)$ or $F \subseteq (P :_\pounds s \vee b)$.

Proof. (1) $\Rightarrow$ (2) Let P be a weakly S-2-absorbing filter of $\pounds$ and assume that $s \in S$ satisfies weakly S-2-absorbing condition. Suppose that $(P :_\pounds a \vee b) \neq (1 :_\pounds a \vee b)$. Since $(1 :_\pounds a \vee b) \subsetneqq (P :_\pounds a \vee b)$, we conclude that there exists $e \in (P :_\pounds a \vee b)$ such that $a \vee b \vee e \neq 1$. Let $z \in (P :_\pounds a \vee b)$. If $a \vee b \vee z \neq 1$, then $s \vee a \vee z \in P$ or $s \vee b \vee z \in P$ by (1), and so $z \in (P :_\pounds s \vee a)$ or $z \in (P :_\pounds s \vee b)$. Now, suppose that $a \vee b \vee z = 1$. Then $1 \neq a \vee b \vee e = (a \vee b \vee e) \wedge (a \vee b \vee z) = (a \vee b) \vee (z \wedge e) \in P$ implies that $(s \vee a) \vee (z \wedge e) = (s \vee a \vee z) \wedge (s \vee a \vee e) \in P$ or $(s \vee b) \vee (z \wedge e) = (s \vee b \vee z) \wedge (s \vee b \vee e) \in P$; hence $s \vee a \vee z \in P$ or $\vee b \vee z \in P$ by Lemma 2.1 (1). Thus $z \in (P :_\pounds s \vee a)$ or $z \in (P :_\pounds s \vee b)$, i.e. $(P :_\pounds a \vee b) \subseteq (P :_\pounds s \vee a)$ or $(P :_\pounds a \vee b) \subseteq (P :_\pounds s \vee b)$.

(2) $\Rightarrow$ (3) Let $a, b \in \pounds$ and F a filter of $\pounds$ such that $\{1\} \neq (a \vee b) \vee F \subseteq P$ and suppose that s has the stated property in (2). Assume that $s \vee a \vee b \notin P$. Since $(a \vee b) \vee F \subseteq P$, we conclude that $F \subseteq (P :_\pounds a \vee b)$ and by (2), $F \subseteq (1 :_\pounds a \vee b)$ or $F \subseteq (P :_\pounds s \vee a)$ or $F \subseteq (P :_\pounds s \vee b)$. If $F \subseteq (1 :_\pounds a \vee b)$, then $(a \vee b) \vee F = \{1\}$ which is impossible. Therefore, either $F \subseteq (P :_\pounds s \vee a)$ or $F \subseteq (P :_\pounds s \vee b)$.

(3) $\Rightarrow$ (1) Let $a, b, c \in \pounds$ such that $1 \neq a \vee b \vee c \in P$. Then $\{1\} \neq (a \vee b) \vee T(\{c\}) \subseteq P$ gives $s \vee a \vee b \in P$ or $T(\{c\}) \subseteq (P :_\pounds s \vee a)$ or $T(\{c\}) \subseteq (P :_\pounds s \vee b)$ by (3) which implies that $s \vee a \vee b \in P$ or $s \vee a \vee c \in P$ or $s \vee b \vee c \in P$, i.e. (1) holds. □

Corollary 3.13. *For proper filter P of $\pounds$, The following assertions are equivalent:*

*(1) **p** is a weakly 2-absorbing filter of $\pounds$;*

(2) For any $a, b \in \pounds$, if $a \vee b \notin P$, then $(P :_\pounds a \vee b) = (1 :_\pounds a \vee b)$ or $(P :_\pounds a \vee b) \subseteq (P :_\pounds a)$ or $(P :_\pounds a \vee b) \subseteq (P :_\pounds b)$;

(3) For any $a, b \in \pounds$ and for any filter F of $\pounds$, if $\{1\} \neq (a \vee b) \vee F \subseteq P$, then $a \vee b \in P$ or $F \subseteq (P :_\pounds a)$ or $F \subseteq (P :_\pounds b)$.

Proof. Take $S = \{0\}$ in Theorem 3.12. □

Lemma 3.14. *Let P be a filter of $\pounds$ and S a join closed subset of $\pounds$ disjoint with P. The following assertions are equivalent:*

(1) P is a weakly S-2-absorbing filter of $\pounds$;

(2) There exists an $s \in S$ such that for any $a, b \in \pounds$, if $\{1\} \neq (a \vee b) \vee F \subseteq P$ for some filter F of $\pounds$, then $s \vee a \vee b \in P$ or $F \subseteq (P :_\pounds s \vee a)$ or $F \subseteq (P :_\pounds s \vee b)$.

Proof. (1) $\Rightarrow$ (2) By the hypothesis, we keep in mind that there exists a fixed $s \in S$ that satisfies the weakly S-2-absorbing condition. Let $a, b \in \pounds$ such that $\{1\} \neq (a \vee b) \vee F \subseteq P$ for some filter F of $\pounds$. Suppose that $s \vee a \vee b \notin P$. Now, it suffices to show that $F \subseteq (P :_\pounds s \vee a)$ or $F \subseteq (P :_\pounds s \vee b)$. Let $f \in F$ (so $a \vee b \vee f \in P$). If $a \vee b \vee f \neq 1$, then $s \vee a \vee f \in P$ or $s \vee b \vee f \in P$ which implies that $F \subseteq (P :_\pounds s \vee a)$ or $F \subseteq (P :_\pounds s \vee b)$. So suppose that $a \vee b \vee f = 1$. Since $(a \vee b) \vee F \neq \{1\}$, we conclude that $1 \neq a \vee b \vee f_1 \in P$ for some $f_1 \in F$; so $s \vee a \vee f_1 \in P$ or $s \vee b \vee f_1 \in P$. Now, we put $f_2 = f \wedge f_1$. Then $1 \neq a \vee b \vee f_2 = a \vee b \vee f_1 \in P$ which gives $s \vee a \vee f_2 \in P$ or $s \vee b \vee f_2 \in P$. We split the proof into three cases.

Case 1: $s \vee a \vee f_1 \in P$ and $s \vee b \vee f_1 \in P$. Since $s \vee a \vee f_2 = (s \vee a \vee f) \wedge (s \vee a \vee f_1) \in P$ or $s \vee b \vee f_2 = (s \vee b \vee f) \wedge (s \vee b \vee f_1) \in P$, we get that $s \vee a \vee f \in P$ or $s \vee b \vee f \in P$ by Lemma 2.1 (1); hence $F \subseteq (P :_\pounds s \vee a)$ or $F \subseteq (P :_\pounds s \vee b)$.

Case 2: $s \vee a \vee f_1 \in P$ and $s \vee b \vee f_1 \notin P$. On the contrary, assume that $s \vee a \vee f \notin P$ and $s \vee b \vee f \notin P$. Then $s \vee a \vee f_2 = (s \vee a \vee f) \wedge (s \vee a \vee f_1) \notin P$ by Lemma 2.1 (1); so $s \vee b \vee f_2 = (s \vee b \vee f) \wedge (s \vee b \vee f_1) \in P$ which is impossible by Lemma 2.1 (1). Thus $s \vee a \vee f \in P$ or $s \vee b \vee f \in P$ and so $F \subseteq (P :_\pounds s \vee a)$ or $F \subseteq P :_\pounds s \vee b)$.

Case 3: $s \vee a \vee f_1 \notin P$ and $s \vee b \vee f_1 \in P$. This proof is similar to that in Case (2) and we omit it.

(2) $\Rightarrow$ (1) Let $x, y, z \in \pounds$ such that $1 \neq x \vee y \vee z \in P$. Then $\{1\} \neq (x \vee y) \vee T(\{z\}) \subseteq P$ gives $s \vee x \vee y \in P$ or $T(\{z\}) \subseteq (P :_\pounds s \vee x)$ or $T(\{z\}) \subseteq (P :_\pounds s \vee y)$ by (2) which implies that $s \vee x \vee y \in P$ or $s \vee x \vee z \in P$ or $s \vee y \vee z \in P$, i.e. (1) holds. □

Lemma 3.15. *Let P be a filter of $\pounds$ and S a join closed subset of $\pounds$ disjoint with P. The following assertions are equivalent:*

(1) P is a weakly S-2-absorbing filter of $\pounds$;

(2) There exists an $s \in S$ such that for any filters F, G of $\pounds$ and $a \in \pounds$, if $\{1\} \neq a \vee (F \vee G) \subseteq P$, then $F \subseteq (P :_\pounds s \vee a)$ or $G \subseteq (P :_\pounds s \vee a)$ or $F \vee G \subseteq (P :_\pounds s)$.

Proof. (1) $\Rightarrow$ (2) Let P be a weakly S-2-absorbing filter of $\pounds$ and assume that $s \in S$ satisfies weakly S-2-absorbing condition. Let F, G be filters of $\pounds$ and $a \in \pounds$ such that $\{1\} \neq a \vee (F \vee G) \subseteq P$. Suppose that $(s \vee a) \vee F \nsubseteq P$. So $s \vee a \vee f \notin P$ for some $f \in F$. We claim that there exists $b \in F$ such that $(a \vee b) \vee G \neq \{1\}$ and $s \vee a \vee b \notin P$. Since $a \vee (F \vee G) \neq \{1\}$, we conclude that $(a \vee f_1) \vee G \neq \{1\}$ for some $f_1 \in F$. Suppose that $s \vee a \vee f_1 \notin P$ or $(a \vee f) \vee G \neq \{1\}$. If $s \vee a \vee f_1 \notin P$, then we put $b = f_1$ and so $s \vee a \vee b \notin P$ and $(a \vee b) \vee G \neq \{1\}$. If $(a \vee f) \vee G \neq \{1\}$, then we put $b = f$ and so $s \vee a \vee b \notin P$ and $(a \vee b) \vee G \neq \{1\}$. Hence, by putting $b = f$ or $b = f_1$, we get the result. Therefore, suppose that $s \vee a \vee f_1 \in P$ and $(a \vee f) \vee G = \{1\}$. It follows that $\{1\} \neq a \vee (f \wedge f_1) \vee G = ((a \vee f_1) \wedge (a \vee f)) \vee G = (a \vee f_1) \vee G \subseteq P$ and $(s \vee a) \vee (f \wedge f_1) = (s \vee a \vee f_1) \wedge (s \vee a \vee f) \notin P$ by Lemma 2.1 (1). So we find $b \in F$ such that $(a \vee b) \vee G \neq \{1\}$ and $s \vee a \vee b \notin P$.

Since $\{1\} \neq (a \vee b) \vee G \subseteq a \vee (F \vee G) \subseteq P$ and $s \vee a \vee b \notin P$, we obtain $G \subseteq (P :_\pounds s \vee a)$ or $G \subseteq (P :_\pounds s \vee b)$ by Lemma 3.14. If $G \subseteq (P :_\pounds s \vee a)$, then we are done. So we may assume that $G \nsubseteq (P :_\pounds s \vee a)$ and so $G \subseteq (P :_\pounds s \vee b)$. Let $c \in F$. If $(a \vee c) \vee G \neq \{1\}$, then by Lemma 3.14, $c \in (P :_\pounds s \vee a)$ or $c \in (P :_\pounds s \vee G)$ since $(s \vee a) \vee G \nsubseteq P$; hence $F \subseteq (P :_\pounds s \vee a)$ or $F \vee G \subseteq (P :_\pounds s)$, i.e. (2) holds. If $(a \vee c) \vee G = \{1\}$, then $\{1\} \neq a \vee (b \wedge c) \vee G = ((a \vee b) \wedge (a \vee c)) \vee G = (a \vee b) \vee G \subseteq P$. Now, Lemma 3.14 gives $(s \vee a) \vee (b \wedge c) \in P$ or $s \vee (b \wedge c) \vee G \subseteq P$ which implies that $b \wedge c \in (P :_\pounds s \vee a)$ or $b \wedge c \in (P :_\pounds s \vee G)$. Now assume that $b \wedge c \in (P :_\pounds s \vee a)$ and $b \wedge c \notin (P :_\pounds s \vee G)$. Consider $\{1\} \neq a \vee (b \wedge c) \vee G = ((a \vee b) \wedge (a \vee c)) \vee G = (a \vee b) \vee G \subseteq P$. By Lemma 3.14, $(s \vee a \vee b) \wedge (s \vee a \vee c) = (s \vee a) \vee (b \wedge c) \in P$ or $s \vee (b \wedge c) \vee G \subseteq P$ since $(s \vee a) \vee G \nsubseteq P$; hence $s \vee a \vee b \in P$ by Lemma 2.1 (1) or $b \wedge c \in (P :_\pounds s \vee G)$, a contradiction. Thus $b \wedge c \in (P :_\pounds s \vee G)$ and so $s \vee (b \wedge c) \vee G \subseteq P$. Let $g \in G$. Then $s \vee (b \wedge c) \vee g = (s \vee b \vee g) \wedge (s \vee c \vee g) \in P$ gives $s \vee c \vee g \in P$ by lemma 2.1 (1) and so $c \in (P :_\pounds s \vee G)$ which implies that $F \subseteq (P :_\pounds s \vee G)$. Therefore, $F \vee G \subseteq (P :_\pounds s)$.

(2) $\Rightarrow$ (1) Let $a, b, c \in \pounds$ such that $1 \neq a \vee b \vee c \in P$. Set $F = T(\{b\})$ and $G = T(\{c\})$. Then $\{1\} \neq a \vee (F \vee G) \subseteq P$ gives $s \vee a \vee b \in (s \vee a) \vee F \subseteq P$ or $s \vee a \vee c \in (s \vee a) \vee G \subseteq P$ or $s \vee b \vee c \in s \vee (F \vee G) \subseteq P$ by (2), as required. □

Proposition 3.16. *Let P be a filter of $\pounds$ and S a join closed subset of $\pounds$ disjoint with P. The following assertions are equivalent:*

(1) P is a weakly S-2-absorbing filter of $\pounds$;

(2) There exists an $s \in S$ such that for any filters F, G, K of $\pounds$, if $\{1\} \neq F \vee G \vee K \subseteq P$, then $F \vee G \subseteq (P :_\pounds s)$ or $F \vee K \subseteq (P :_\pounds s)$ or $G \vee K \subseteq (P :_\pounds s)$.

Proof. (1) $\Rightarrow$ (2) Let P be a weakly S-2-absorbing filter of $\pounds$ and assume that $s \in S$ satisfies weakly S-2-absorbing condition. Let F, G, K be filters of $\pounds$ such that $\{1\} \neq F \vee G \vee K \subseteq P$; so $\{\{1\} \neq g \vee (F \vee K) \subseteq P$ for some $g \in G$. By Lemma 3.15, $(s \vee g) \vee F \subseteq P$ or $(s \vee g) \vee G \subseteq P$ or $s \vee (F \vee K) \subseteq P$. If $s \vee (F \vee K) \subseteq P$, then we are done and so assume that $s \vee (F \vee K) \nsubseteq P$. Therefore, we have either $(s \vee g) \vee F \subseteq P$ or $(s \vee g) \vee K \subseteq P$. We claim that either $s \vee (F \vee G) \subseteq P$ or $s \vee (G \vee K) \subseteq P$. Let $g_1 \in G$. If $\{1\} \neq g_1 \vee (F \vee K) \subseteq P$, then by Lemma 3.15, $(s \vee g_1) \vee F \subseteq P$ or $(s \vee g_1) \vee K \subseteq P$ since $s \vee (F \vee K) \nsubseteq P$ which implies that $g_1 \in (P :_\pounds s \vee F)$ or $g_1 \in (P :_\pounds s \vee K)$. It follows that $F \vee G \subseteq (P :_\pounds s)$ or $G \vee K \subseteq (P :_\pounds s)$, i.e. we get the claim. Now let $g_1 \vee (F \vee K) = \{1\}$. Since $\{1\} \neq (g \wedge g_1) \vee (F \vee K) = (g_1 \vee (F \vee K)) \wedge (g \vee (F \vee K)) = g \vee (F \vee K) \subseteq P$, we conclude that $s \vee (g \wedge g_1) \vee F \subseteq P$ or $s \vee (g \wedge g_1) \vee K \subseteq P$ by Lemma 3.15. , we split the proof into four cases.

Case 1: $(s \vee g) \vee F \subseteq P$ and $s \vee (g \wedge g_1) \vee F \subseteq P$.

Since $s \vee (g \wedge g_1) \vee f = (s \vee g \vee f) \wedge (s \vee g_1 \vee f) \in P$ for all $f \in F$, we conclude that $s \vee g_1 \vee f \in P$ by Lemma 2.1 (1) which implies that $(s \vee g_1) \vee F \subseteq P$; hence $s \vee (F \vee G) \subseteq P$.

Case 2: $(s \vee g) \vee K \subseteq P$ and $s \vee (g \wedge g_1) \vee K \subseteq P$.

Since $s \vee (g \wedge g_1) \vee k = (s \vee g \vee k) \wedge (s \vee g_1 \vee k) \in P$ for all $k \in K$, we conclude that $s \vee g_1 \vee k \in P$ by Lemma 2.1 (1) which implies that $(s \vee g_1) \vee K \subseteq P$; hence $s \vee (G \vee K) \subseteq P$.

Case 3: $(s \vee g) \vee F \subseteq P$, $(s \vee g) \vee K \nsubseteq P$, $s \vee (g \wedge g_1) \vee K \subseteq P$ and $s \vee (g \wedge g_1) \vee F \nsubseteq P$.

Since $(s \vee g) \vee K \nsubseteq P$, we conclude that $s \vee g \vee k \notin P$ for some $k \in K$. Then by the hypothesis, $s \vee (g \wedge g_1) \vee k = (s \vee g \vee k) \wedge (s \vee g_1 \vee k) \in P$ which implies that $s \vee g \vee k \in P$ by Lemma 2.1 (1) and this is not possible. Hence since $(s \vee g) \vee F \subseteq P$ or $(s \vee g) \vee K \subseteq P$ or $s \vee (g \wedge g_1) \vee F \subseteq P$ or $s \vee (g \wedge g_1) \vee K \subseteq P$, there must be any one of the following holds:

(i) $(s \vee g) \vee K \subseteq P$ and $s \vee (g \wedge g_1) \vee K \subseteq P$ and $s \vee (g \wedge g_1) \vee F \nsubseteq P$, then $g_1 \in (P :_\pounds s \vee K)$; hence $G \vee K \subseteq (P :_\pounds s)$.

(ii) $(s \vee g) \vee F \subseteq P$ and $(s \vee g) \vee K \nsubseteq P$ and $s \vee (g \wedge g_1) \vee F \subseteq P$, then $g_1 \in (P :_\pounds s \vee F)$; hence $G \vee F \subseteq (P :_\pounds s)$.

Case 4: $s \vee (g \wedge g_1) \vee F \subseteq P$, $s \vee (g \wedge g_1) \vee K \nsubseteq P$, $(s \vee g) \vee K \subseteq P$ and $(s \vee g) \vee F \nsubseteq P$. By an argument like that in the Case (3), we get $g_1 \in (P :_\pounds s \vee F)$ or $g_1 \in (P :_\pounds s \vee K)$. Therefore $F \vee G \subseteq (P :_\pounds s)$ or $G \vee K \subseteq (P :_\pounds s)$.

(2) $\Rightarrow$ (1) Let $a, b, c \in \pounds$ such that $1 \neq a \vee b \vee c \in P$. Set $F = T(\{b\})$, $G = T(\{b\})$ and $K = T(\{c\})$. Then $\{1\} \neq F \vee G \vee K \subseteq P$ gives $s \vee a \vee b \in s \vee (F \vee G) \subseteq P$ or $s \vee a \vee c \in s \vee (F \vee K \subseteq P$ or $s \vee b \vee c \in s \vee (G \vee K) \subseteq P$ by (2), as required. □

The next theorem gives a more explicit description of the weakly S-2-absorbing filters of $\pounds$.

Theorem 3.17. *Let P be a filter of $\pounds$ and S a join closed subset of $\pounds$ disjoint with P. The following assertions are equivalent:*

(1) P is a weakly S-2-absorbing filter of $\pounds$;

(2) There exists an $s \in S$ such that for any $a, b \in \pounds$, if $\{1\} \neq (a \vee b) \vee F \subseteq P$ for some filter F of $\pounds$, then $s \vee a \vee b \in P$ or $F \subseteq (P :_\pounds s \vee a)$ or $F \subseteq (P :_\pounds s \vee b)$.

(3) There exists an $s \in S$ such that for any filters F, G of $\pounds$ and $a \in \pounds$, if $\{1\} \neq a \vee (F \vee G) \subseteq P$, then $F \subseteq (P :_\pounds s \vee a)$ or $G \subseteq (P :_\pounds s \vee a)$ or $F \vee G \subseteq (P :_\pounds s)$.

(4) There exists an $s \in S$ such that for any filters F, G, K of $\pounds$, if $\{1\} \neq F \vee G \vee K \subseteq P$, then $F \vee G \subseteq (P :_\pounds s)$ or $F \vee K \subseteq (P :_\pounds s)$ or $G \vee K \subseteq (P :_\pounds s)$.

Proof. This is a direct consequence Lemma 3.14, Lemma 3.15 and Proposition 3.16. □

4. Further results

We continue in this section with the investigation of the stability of weakly S-2-absorbing filters in various lattice-theoretic constructions.

Proposition 4.1. *Let S be a join closed subset of $\pounds$ and P a weakly S-2-absorbing filter of $\pounds$ such that $P \cap S = \emptyset$. If Q is a filter of $\pounds$ such that $Q \cap S \neq \emptyset$, then $P \vee Q$ is a weakly S-2-absorbing filter of $\pounds$.*

Proof. Since $(P \vee Q) \cap S \subseteq P \cap S = \emptyset$, we conclude that $P \vee Q) \cap S = \emptyset$. Consider $t \in Q \cap S$. Let $a, b, c \in \pounds$ such that $1 \neq a \vee b \vee c \in P \vee Q \subseteq P$. Then there exists $s \in S$ such that $s \vee a \vee b \in P$ or $s \vee a \vee c \in P$ or $s \vee b \vee c \in P$ which gives $s \vee t \vee a \vee b \in P \vee Q$ or $s \vee t \vee a \vee c \in P \vee Q$ or $s \vee t \vee b \vee c \in P \vee Q$, where $s \vee t \in S$, i.e. $P \vee Q$ is a weakly S-2-absorbing filter of $\pounds$. □

Proposition 4.2. *Suppose that S is a join closed subset of $\pounds$. Then the following assertions are equivalent:*

(1) Every weakly S-2-absorbing filter of $\pounds$ is prime;

(2) $\pounds$ is a $\pounds$-domain and every S-2-absorbing filter of $\pounds$ is prime.

Proof. (1) ⇒ (2) Since $\{1\}$ is a weakly S-2-absorbing filter, we conclude that it is a prime filter by (1) which gives $\pounds$ is a $\pounds$-domain. Finally, since every S-2-absorbing filter of $\pounds$ is weakly S-2-absorbing, we have P is prime by (1).

(2) ⇒ (1) Let P be a weakly S-2-absorbing filter of $\pounds$. It suffices to show that P is an S-2-absorbing filter. Let $a, b, c \in \pounds$ such that $a \vee b \vee c \in P$. If $a \vee b \vee c \neq 1$, then there exists $s \in S$ such that $s \vee a \vee b \in P$ or $s \vee a \vee c \in P$ or $s \vee b \vee c \in P$. If $a \vee b \vee c = 1$, then $a = 1$ or $b = 1$ or $c = 1$; so $s \vee a \vee b = 1 \in P$ or $s \vee a \vee c = 1 \in P$ or $s \vee b \vee c = 1 \in P$ for every $s \in S$. Therefore, every weakly S-2-absorbing filter of $\pounds$ is prime by (2). □

Theorem 4.3. *Let $f : \pounds \to \pounds'$ be a lattice homomorphism such that $f(1) = 1$ and S a join closed subset of $\pounds$. The following hold:*

(1) Let $\pounds$ be a complemented lattice. If f is a epimorphism and P is a weakly S-2-absorbing filter with $\mathrm{Ker}(f) \subseteq P$, then $f(P)$ is a weakly $f(S)$-2-absorbing filter of $\pounds'$;

(2) If f is a monomorphism and P' is a weakly $f(S)$-2-absorbing filter of $\pounds'$, then $P = f^{-1}(P')$ is a weakly S-2-absorbing filter of $\pounds$.

Proof. (1) Clearly, $f(S)$ is a join closed subset of $\pounds'$. Let $c \in f(S) \cap f(P)$. Then $c = f(p) = f(s)$ for some $p \in P$ and $s \in S$. By assumption, there exists $p' \in \pounds$ such that $p \vee p' = 1$ and $p \wedge p' = 0$ which gives $f(s \vee p') = f(p) \vee f(p') = 1$; hence $s \vee p' \in \mathrm{Ker}(f) \subseteq P$. Since $s = s \vee (p \wedge p') = (s \vee p') \wedge (s \vee p) \in P$, we conclude that $s \in S \cap P$, a contradiction. Thus $f(S) \cap f(P) = \emptyset$. Let $x, y, z \in \pounds'$ such that $1 \neq x \vee y \vee z \in f(P)$. Then there exist $a, b, c \in \pounds$ such that $x = f(a)$, $y = f(b)$, $z = f(c)$ and $1 \neq f(a \vee b \vee c) = x \vee y \vee z \in f(P)$ (so $a \vee b \vee c \neq 1$) which implies that $f(a \vee b \vee c) = f(q)$ for some $q \in P$. By the hypothesis, $q \vee q' = 1$ and $q \wedge q' = 0$ for some $q' \in \pounds$. Since $f(a \vee b \vee c \vee q') = 1$, we conclude that $a \vee b \vee c \vee q' \in \mathrm{Ker}(f) \subseteq P$; hence $1 \neq a \vee b \vee c = (a \vee b \vee c) \vee (q \wedge q') = (a \vee b \vee c \vee q) \wedge (a \vee b \vee c \vee q') \in P$, as P is a filter. This implies that $s \vee a \vee b \in P$ or $s \vee a \vee c \in P$ or $s \vee b \vee c \in P$ for some $s \in S$. It means that $f(s) \vee x \vee y \in f(P)$ or $f(s) \vee x \vee z \in f(P)$ or $f(s) \vee y \vee z \in f(P)$. Therefore, $f(P)$ is a weakly $f(S)$-2-absorbing filter of $\pounds'$.

(2) By assumption, there exists $s \in S$ such that for all $x, y, z \in \pounds'$, $x \vee y \vee z \in P'$ implies $f(s) \vee x \vee y \in P'$ or $f(s) \vee x \vee z \in P'$ or $f(s) \vee y \vee z \in P'$. Clearly, $P \cap S = \emptyset$. Let $a, b, c \in \pounds$ such that $1 \neq a \vee b \vee c \in P$. Since $\text{Ker}(f) = \{1\}$ by Lemma 2.3 (2), we conclude that $1 \neq f(a \vee b \vee c) = f(a) \vee f(b) \vee f(c) \in P'$; so $f(s) \vee f(a) \vee f(b) = f(s \vee a \vee b) \in P'$ or $f(s) \vee f(a) \vee f(c) = f(s \vee a \vee c) \in P'$ or $f(s) \vee f(b) \vee f(c) = f(s \vee b \vee c) \in P'$. Hence, $s \vee a \vee b \in P$ or $s \vee a \vee c \in P$ or $s \vee b \vee c \in P$, and so $P = f^{-1}(P')$ is a weakly S-2-absorbing filter of $\pounds$. □

Corollary 4.4. *Let S be a join closed subset of $\pounds$. If $\pounds$ is a sublattice of $\pounds'$ and G' is a weakly S-2-absorbing filter of $\pounds'$, then $G' \cap \pounds$ is a weakly S-2-absorbing filter of $\pounds$.*

Proof. It suffices to apply Theorem 4.3 (2) to the natural injection $\iota : \pounds \to \pounds'$ since $\iota^{-1}(G') = G' \cap \pounds$. □

Let F be a filter of $\pounds$ and S a join closed subset of $\pounds$ disjoint with F. It is clear that $S_Q = \{s \wedge F : s \in S\}$ is a join closed subset of $\pounds/F$.

Theorem 4.5. *Let S be a join closed subset of $\pounds$, F and G are two filters of $\pounds$ with $F \subseteq G$. The following hold:*

(1) Let $\pounds$ be a complemented lattice. If G is a weakly S-2-absorbing filter of $\pounds$, then G/F is a weakly S_Q-2-absorbing filter of $\pounds/F$;

(2) If G/F is a weakly S_Q-2-absorbing filter of $\pounds/F$ and F is a weakly S-2-absorbing filter of $\pounds$, then G is a weakly S-2-absorbing filter of $\pounds$.

Proof. (1) Assume that $f : \pounds \to \pounds/F$ such that $f(a) = a \wedge F$ and let $x, y \in \pounds$. Then $f(x \vee y) = (x \vee y) \wedge F = (x \wedge F) \vee_Q (y \wedge F) = f(x) \vee_Q f(y)$. Similarly, $f(x \wedge y) = f(x) \wedge_Q f(y)$. So f is a lattice homomorphism from $\pounds$ onto $\pounds/F$ and $f(1) = 1 \wedge F = 1_{\pounds/F}$. Suppose that G is a weakly S-2-absorbing filter of $\pounds$. Since $\text{Ker}(f) = F \subseteq G$ and f is onto, we conclude that $f(G) = G/F$ (see [8, Lemma 3.4]) is a S_Q-2-absorbing filter of $\pounds/F$ by Theorem 4.3 (1).

(2) Let $a, b, c \in \pounds$ such that $1 \neq a \vee b \vee c \in G$. Then $(a \wedge F) \vee_Q (b \wedge F) \vee_Q (c \wedge F) = (a \vee b \vee c) \wedge F \in G/F$. If $(a \vee b \vee c) \wedge F \neq 1_{\pounds/F} = 1 \wedge F$, then G/F is a weakly S_Q-2-absorbing gives there exists $s \in S$ such that $(s \wedge F) \vee_Q (a \wedge F) \vee_Q (b \wedge F) = (s \vee a \vee b) \wedge F \in G/F$ or $(s \wedge F) \vee_Q (a \wedge F) \vee_Q (c \wedge F) = (s \vee a \vee c) \wedge F \in G/F$ or $(s \wedge F) \vee_Q (b \wedge F) \vee_Q (c \wedge F) = (s \vee b \vee c) \wedge F \in G/F$ which implies that $s \vee a \vee b) \in G$ or $s \vee a \vee c \in G$ or $s \vee b \vee c \in G$. If $(a \vee b \vee c) \wedge F = 1 \wedge F$, then there exist $f_1, f_2 \in F$ such that $(a \vee b \vee c) \wedge f_1 = 1 \wedge f_2 = f_2 \in F$; so $1 \neq a \vee b \vee c \in F$ by Lemma 2.1 (1) which gives there is an element $t \in S$ such that $t \vee a \vee b \in F \subseteq G$ or $t \vee a \vee c \in F \subseteq G$ or $t \vee b \vee c \in F \subseteq G$. This shows that G is a weakly S-2-absorbing filter of $\pounds$. □

Theorem 4.6. *Let $\pounds = \pounds_1 \times \pounds_2$ be a decomposable lattice and $S = S_1 \times S_2$, where S_i is a join closed subset of $\pounds_i$. Suppose that $P = P_1 \times P_2$, where $P_1 \neq \{1\}$ is a filter of $\pounds_1$ and $P_2 \neq \{1\}$ is a filter of $\pounds_2$. Then the following assertions are equivalent:*

(1) P is a weakly S-2-absorbing filter of $\pounds$;

(2) P_1 is a weakly S_1-2-absorbing filter of $\pounds_1$ and $P_2 \cap S_2 \neq \emptyset$ or P_2 is a weakly S_2-2-absorbing filter of $\pounds_2$ and $P_1 \cap S_1 \neq \emptyset$ or P_1 is a weakly S_1-prime filter of $\pounds_1$ and P_2 is a weakly S_2-prime filter of $\pounds_2$.

Proof. (1) ⇒ (2) Let P be a weakly S-2-absorbing filter of $\pounds$ and assume that $s = (s_1, s_2) \in S$ satisfies weakly S-2-absorbing condition. As $P \cap S = \emptyset$, we get either $P_1 \cap S_1 = \emptyset$ or $P_2 \cap S_2 = \emptyset$. If $P_1 \cap S_1 \neq \emptyset$, we will show that P_2 is a weakly S_2-2-absorbing filter of $\pounds_2$. Let $1 \neq a \vee b \vee c \in P_2$ for some $a, b, c \in \pounds_2$. Then $(1, 1) \neq (1, a) \vee_c (1, b) \vee_c (1, c) = (1, a \vee b \vee c) \in P$ gives $s \vee_c (1, a) \vee_c (1, b) = (1, s_2 \vee a \vee b) \in P$ or $s \vee_c (1, a) \vee_c (1, c) = (1, s_2 \vee a \vee c) \in P$ or $s \vee_c (1, b) \vee_c (1, c) = (1, s_2 \vee b \vee c) \in P$. This shows that $s_2 \vee a \vee b \in P_2$ or $s_2 \vee a \vee c \in P_2$ or $s_2 \vee b \vee c \in P_2$. Hence, P_2 is a weakly S_2-2-absorbing filter of $\pounds_2$. Similarly, if $S_2 \cap P_2 \neq \emptyset$, then P_1 is a weakly S_1-2-absorbing filter of $\pounds_1$.

Now, assume that $S_1 \cap P_1 = \emptyset = S_2 \cap P_2$. We will show that P_1 is a weakly S_1-prime filter of $\pounds_1$ and P_2 is a weakly S_2-prime filter of $\pounds_2$. Suppose that P_1 is not a weakly S_1-prime filter of $\pounds_1$. Then there exist $x, y \in \pounds_1$ such that $1 \neq x \vee y \in P_1$ but $s_1 \vee x, s_1 \vee y \notin P_1$. Since $S_2 \cap P_2 = \emptyset$, we conclude that

$s_2 \notin P_2$. Then $(1,1) \neq (x,0) \vee_c (0,1) \vee_c (y,s_2) = (x \vee y, 1) \in P$ gives $s \vee_c (x,0) \vee_c (0,1) = (s_1 \vee x, 1) \in P$ or $s \vee_c (x,0) \vee_c (y,s_2) = (s_1 \vee x \vee y, s_2) \in P$ or $s \vee_c (0,1) \vee_c (y,s_2) = (s_1 \vee y, 1) \in P$; so $s_1 \vee x \in P_1$ or $s_2 \in P_2$ or $s_1 \vee y \in P_1$ which is a contradiction. Therefor, P_1 is a weakly S_1-prime filter of $\pounds_2$. Similarly, P_2 is a weakly S_2-prime filter of $\pounds_2$.

(2) $\Rightarrow$ (1) Let $P_1 \cap S_1 \neq \emptyset$ and P_2 be a weakly S_2-2-absorbing filter of $\pounds_2$. At first, note that $P \cap S = \emptyset$. Let $(1,1) \neq (a,x) \vee_c (b,y) \vee_c (c,z) = (a \vee b \vee c, x \vee y \vee z) \in P$ for some $(a,x),(b,y),(c,z) \in \pounds$. Since $P_1 \cap S_1 \neq \emptyset$, there exists $s_1 \in S_1$ such that $s_1 \vee v \in P_1$ for all $v \in \pounds_1$. Also, there exists $s_2 \in S_2$ satisfying P_2 to be a weakly S_2-2-absorbing filter of $\pounds_2$. Now, put $s = (s_1, s_2) \in S$. If $x \vee y \vee z \neq 1$, then P_2 is a weakly S_2-2-absorbing filter gives $s_2 \vee x \vee y \in P_2$ or $s_2 \vee x \vee z \in P_2$ or $s_2 \vee y \vee z \in P_2$. This shows that $s \vee_c (a,x) \vee_c (b,y) \in P$ or $s \vee_c (a,x) \vee_c (c,z) \in P$ or $s \vee_c (c,z) \vee_c (b,y) \in P$. Now, assume that $x \vee y \vee z = 1$. Since $P_2 \neq \{1\}$, there exists $p_2 \in P_2$ such that $p_2 \neq 1$. As $1 \neq (x \wedge p_2) \vee_c (y \wedge p_2) \vee_c (z \wedge p_2) = p_2 \in P_2$, we conclude that $s_2 \vee (x \wedge p_2) \vee_c (y \wedge p_2) = (s_2 \vee p_2) \wedge (s_2 \vee x \vee y) \in P_2$ or $s_2 \vee (x \wedge p_2) \vee_c (z \wedge p_2) = (s_2 \vee p_2) \wedge (s_2 \vee x \vee z) \in P_2$ or $s_2 \vee (y \wedge p_2) \vee_c (z \wedge p_2) = (s_2 \vee p_2) \wedge (s_2 \vee y \vee z) \in P_2$; hence $s_2 \vee x \vee y \in P_2$ or $s_2 \vee x \vee z \in P_2$ or $s_2 \vee y \vee z \in P_2$ by Lemma 2.1 (1). This implies that $s \vee_c (a,x) \vee_c (b,y) \in P$ or $s \vee_c (a,x) \vee_c (c,z) \in P$ or $s \vee_c (c,z) \vee_c (b,y) \in P$. Hence, P is a weakly S-2-absorbing filter of $\pounds$. If $P_2 \cap S_2 \neq \emptyset$ and P_1 is a weakly S_1-2-absorbing filter of $\pounds_1$, similar argument shows that P is an S-2-absorbing filter.

Now, suppose that for each $i = 1, 2$, P_i is a weakly S_i-prime filter of $\pounds_i$. Let $(1,1) \neq (a,x) \vee_c (b,y) \vee_c (c,z) = (a \vee b \vee c, x \vee y \vee z) \in P$ for some $(a,x),(b,y),(c,z) \in \pounds$. If $1 \neq a \vee b \vee c \in P_1$, then there exists a fixed $s_1 \in S_1$ such that $s_1 \vee a \in P_1$ or $s_1 \vee b \in P_1$ or $s_1 \vee c \in P_1$. So Suppose that $a \vee b \vee c = 1$. Consider $1 \neq p_1 \in P_1$. Then $1 \neq (p_1 \wedge a) \vee (p_1 \wedge b) \vee (p_1 \wedge c) = p_1 \in P_1$ gives $s_1 \vee (a \wedge p_1) = (s_1 \vee p_1) \wedge (s_1 \vee a) \in P_1$ or $s_1 \vee (b \wedge p_1) = (s_1 \vee p_1) \wedge (s_1 \vee b) \in P_1$ or $s_1 \vee (c \wedge p_1) = (s_1 \vee p_1) \wedge (s_1 \vee c) \in P_1$ which implies that $s_1 \vee a \in P_1$ or $s_1 \vee b \in P_1$ or $s_1 \vee c \in P_1$ by Lemma 2.1 (1). Similarly, there exists $s_2 \in S_2$ such that $s_2 \vee x \in P_2$ or $s_2 \vee y \in P_2$ or $s_2 \vee z \in P_2$. Put $s = (s_1, s_2) \in S$. Without loss of generality, we may assume that $s_1 \vee a \in P_1$ and $s_2 \vee z \in P_2$. Then $s \vee_c (a,x) \vee_c (c,z) \in P$. Therefore, P is an S-2-absorbing filter of $\pounds$. □

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