J. Algebra Comb. Discrete Appl.
12(3) • 275–288

Received: 15 April 2024
Accepted: 30 June 2025

Journal of Algebra Combinatorics Discrete Structures and Applications

Local and 2-local $\frac{1}{2}$-derivation on naturally graded non-Lie p-filiform Leibniz algebras

Research Article

Bakhtiyor Yusupov

Abstract: This paper is devoted to study local and 2-local $\frac{1}{2}$-derivation on p-filiform Leibniz algebras. We prove that p-filiform Leibniz algebras as a rule admit local(2-local) $\frac{1}{2}$-derivations which are not $\frac{1}{2}$-derivations.

2020 MSC: 17A36, 17B20, 17B40.

Keywords: Leibniz algebra, p-filiform Leibniz algebras, $\frac{1}{2}$-derivations, local $\frac{1}{2}$-derivations, 2-local $\frac{1}{2}$-derivations

1. Introduction

The notion of δ-derivations was initiated by V. Filippov for Lie algebras in [16, 17]. The space of δ-derivations includes usual derivations ($\delta = 1$), anti-derivations ($\delta = -1$) and elements from the centroid. In [17] it was proved that prime Lie algebras, as a rule, do not have nonzero δ-derivations (in case of $\delta \neq 1, -1, 0, \frac{1}{2}$), and all $\frac{1}{2}$-derivations of an arbitrary prime Lie algebra A over the field $\mathbb{F}$ ($\frac{1}{6} \in \mathbb{F}$) with a non-degenerate symmetric invariant bilinear form were described. It was proved that if A is a central simple Lie algebra over a field of characteristic $p \neq 2, 3$ with a non-degenerate symmetric invariant bilinear form, then any $\frac{1}{2}$-derivation D has the form $D(x) = \lambda x$ for some $\lambda \in \mathbb{F}$. In [18], δ-derivations were investigated for prime alternative and non-Lie Malcev algebras, and it was proved that alternative and non-Lie Malcev algebras with certain restrictions on the ring of operators F have no non-trivial δ-derivation.

Local derivations are useful tools in studying the structure of rings and algebras, where there are still many related unsolved problems. R.V. Kadison, D.R. Larson and A.R. Sourour first introduced the

Bakhtiyor Yusupov (Corresponding Author); V.I.Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, 9, Univesity street, 100174, Tashkent, Uzbekistan
Department of Algebra and Mathematical Engineering, Urgench State University, H. Alimdjan street, 14, Urgench 220100, Uzbekistan (email: baxtiyor_yusupov_93@mail.ru)

notion of local derivations on algebras in their remarkable paper [20, 24]. Since then many researches have been studying local derivations of different types of algebras (e.g. see [1–4, 6, 7, 10, 21]). In [6] the authors proved that every local derivation on a finite-dimensional semisimple Lie algebra $\mathcal{L}$ over an algebraically closed field of characteristic zero is a derivation. In [10] local derivations of solvable Lie algebras are investigated and it is shown that in the class of solvable Lie algebras, there exist algebras which admit local derivations which are not ordinary derivation and also algebras for which every local derivation is a derivation. Moreover, it is proved that every local derivation on a finite-dimensional solvable Lie algebra with model nilradical and maximal dimension of complementary space is a derivation. In [21], the authors proved that every local derivations on solvable Lie algebras whose nilradical has maximal rank is a derivation. In [3], the authors proved that every local derivation on the conformal Galilei algebra is a derivation. The results of the paper [7] show that p-filiform Leibniz algebras as a rule admit local derivations which are not derivations. In [2] the authors proved proved that direct sum of null-filiform nilpotent Leibniz algebras as a rule admit local derivations which are not derivations.

We note that aforementioned algerbas are finite-dimensional algebras. In the infinite-dimensional case, the authors of [8, 14, 27] proved that every local derivation on some class of the locally simple Lie algebras, generalized Witt algebras, Witt algebras, the Witt algebras over a field of prime characteristic is a derivation.

A similar notion, which characterizes non-linear generalizations of automorphisms and derivations, was introduced by P. Šemrl in [25] as 2-local automorphisms (respectively, 2-local derivations).

Several papers have been devoted to similar notions and corresponding problems for 2-local derivations and automorphisms of Lie algebras [5, 8, 9, 11–13, 19, 26, 28]. Namely, in [5] it is proved that every 2-local derivation on the semi-simple Lie algebras is a derivation and that each finite-dimensional nilpotent Lie algebra, with dimension larger than two admits 2-local derivation, which is not a derivation. Let us present a list of finite or infinite-dimensional Lie algebras for which all 2-local derivations are derivations: finite-dimensional semi-simple Lie algebras over an algebraically closed field of characteristic zero; infinite-dimensional Witt algebras over an algebraically closed field of characteristic zero; locally finite split simple Lie algebras over a field of characteristic zero; Virasoro algebras; Virasoro-like algebra; the Schrodinger-Virasoro algebra; Jacobson-Witt algebras; planar Galilean conformal algebras.

Investigation of local and 2-local δ-derivations on Lie algebras was initiated in [22] by A. Khudoyberdiyev and B. Yusupov. Namely, in [22] it is proved we introduce the notion of local and 2-local δ-derivations and describe local and 2- local $\frac{1}{2}$-derivation of finite-dimensional solvable Lie algebras with filiform, Heisenberg, abelian nilradicals. Moreover, we give the description of local $\frac{1}{2}$-derivation of oscillator Lie algebras, conformal perfect Lie algebras, and Schrödinger algebras. Similar problem [23] U. Mamadaliyev, A. Sattarov and B. Yusupov investigated local and 2-local $\frac{1}{2}$-derivations on Leibniz algebras. They proved that any local $\frac{1}{2}$-derivation on the solvable Leibniz algebras with model or abelian nilradicals, whose the dimension of complementary space is maximal is a $\frac{1}{2}$-derivation. They proved that solvable Leibniz algebras with abelian nilradicals, which have 1-dimension complementary space is a $\frac{1}{2}$-derivation. Moreover, similar problem concerning 2-local $\frac{1}{2}$-derivations of such algebras are investigated and an example of solvable Leibniz algebra given such that any 2-local $\frac{1}{2}$-derivation on it is a $\frac{1}{2}$-derivation, but which admit 2-local $\frac{1}{2}$-derivations which are not $\frac{1}{2}$-derivations. B. Yusupov, V. Vaisova and T. Madrakhimov proved similar results concerning local $\frac{1}{2}$-derivations of naturally graded quasi-filiform Leibniz algebras of type I in their recent paper [29]. They proved that quasi-filiform Leibniz algebras of type I, as a rule, admit local $\frac{1}{2}$-derivations which are not $\frac{1}{2}$-derivations.

In the present paper we study $\frac{1}{2}$-derivation, local $\frac{1}{2}$-derivation and 2-local $\frac{1}{2}$-derivation p-filiform Leibniz algebras. In Section 3 we describe the $\frac{1}{2}$-derivation p-filiform Leibniz algebras. In Section 4 we describe the local $\frac{1}{2}$-derivation p-filiform Leibniz algebras. We show that in section 4 we describe p-filiform Leibniz algebras as a rule admit local(2-local) $\frac{1}{2}$-derivation which are not $\frac{1}{2}$-derivation.

2. Preliminaries

In this section we give some necessary definitions and preliminary results.

Definition 2.1. *A vector space with bilinear bracket $(\mathcal{L}, [\cdot,\cdot])$ is called a Leibniz algebra if for any $x, y, z \in \mathcal{L}$ the so-called Leibniz identity*

$$\big[x,[y,z]\big] = \big[[x,y],z\big] - \big[[x,z],y\big],$$

holds.

Let $\mathcal{L}$ be a Leibniz algebra. For a Leibniz algebra $\mathcal{L}$ consider the following central lower and derived sequences:

$$\mathcal{L}^1 = \mathcal{L}, \quad \mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}^1], \quad k \geq 1,$$

$$\mathcal{L}^{[1]} = \mathcal{L}, \quad \mathcal{L}^{[s+1]} = [\mathcal{L}^{[s]}, \mathcal{L}^{[s]}], \quad s \geq 1.$$

Definition 2.2. *A Leibniz algebra $\mathcal{L}$ is called nilpotent (respectively, solvable), if there exists $p \in \mathbb{N}$ ($q \in \mathbb{N}$) such that $\mathcal{L}^p = 0$ (respectively, $\mathcal{L}^{[q]} = 0$).The minimal number p (respectively, q) with such property is said to be the index of nilpotency (respectively, of solvability) of the algebra $\mathcal{L}$.*

Now let us define a naturally graduation for a nilpotent Leibniz algebra.

Definition 2.3. *Given a nilpotent Leibniz algebra $\mathcal{L}$, put $\mathcal{L}_i = \mathcal{L}^i/\mathcal{L}^{i+1}$, $1 \leq i \leq n-1$, and $gr(\mathcal{L}) = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_{n-1}$. Then $[\mathcal{L}_i, \mathcal{L}_j] \subseteq \mathcal{L}_{i+j}$ and we obtain the graded algebra $gr(\mathcal{L})$. If $gr(\mathcal{L})$ and L are isomorphic, then we say that an algebra $\mathcal{L}$ is naturally graded.*

Now we define the notion of characteristic sequence, which is one of the important invariants. For a finite-dimensional nilpotent Leibniz algebra N and for the matrix of the linear operator R_x denote by $C(x)$ the descending sequence of its Jordan blocks' dimensions. Consider the lexicographical order on the set $C(N) = \{C(x) \mid x \in N\}$.

Definition 2.4. *The sequence*

$$\left(\max_{x \in N \setminus N^2} C(x)\right)$$

is said to be the characteristic sequence of the nilpotent Leibniz algebra N.

Definition 2.5. *A Leibniz algebra $\mathcal{L}$ is called p-filiform, if the characteristic sequence is $C(\mathcal{L}) = (n - p, \underbrace{1, \ldots, 1}_{p})$.*

Definition 2.6. *Let $(\mathfrak{L}, [-,-])$ be an algebra with a multiplication $[-,-]$, D be a linear map and D be a bilinear map. Then D is a δ-derivation if it satisfies*

$$D[x,y] = \delta\big([D(x), y] + [x, D(y)]\big).$$

Note that 1-derivation is a usual derivation and (-1)-derivation is called anti-derivation. If D_1 and D_2 are δ_1 and δ_2-derivations, respectively, then their commutator $[D_1, D_2] = D_1D_2 - D_2D_1$ ia a $\delta_1\delta_2$ derivation. Thus, the set of all δ-derivations of a Lie algebra $\mathcal{L}$ is a Lie algebra with respect to the commutator. The set of all δ-derivations, we denote by $Der_\delta(\mathcal{L})$.

Definition 2.7. *A linear operator Δ is called a local δ-derivation, if for any $x \in \mathcal{L}$, there exists a δ-derivation $D_x : \mathcal{L} \to \mathcal{L}$ (depending on x) such that $\Delta(x) = D_x(x)$. The set of all local δ-derivations on $\mathcal{L}$ we denote by* $\mathrm{LocDer}_\delta(\mathcal{L})$.

Definition 2.8. *A map $\nabla : \mathcal{L} \to \mathcal{L}$ (not necessary linear) is called a 2-local δ-derivation, if for any $x, y \in \mathcal{L}$, there exists a δ-derivation $D_{x,y} \in \mathrm{Der}_\delta(\mathcal{L})$ such that*

$$\nabla(x) = D_{x,y}(x), \quad \nabla(y) = D_{x,y}(y).$$

It should be noted that 2-local δ-derivation is not necessary linear, but for any $x \in \mathfrak{L}$ and for any scalar λ, we have that

$$\nabla(\lambda x) = D_{x,\lambda x}(\lambda x) = \lambda D_{x,\lambda x}(x) = \lambda \nabla(x).$$

In this work, we focus on investigating local and 2-local $\frac{1}{2}$-derivations. Note that the main example of $\frac{1}{2}$-derivations is the multiplication by an element from the ground field, i.e., $D(x) = \lambda x$ for all $x \in \mathfrak{L}$. Such kind of $\frac{1}{2}$-derivations are called trivial $\frac{1}{2}$-derivations.

Since we shall consider naturally graded non-Lie p-filiform nilradical we give its classication.

Theorem 2.9. *[15] An arbitrary n-dimensional naturally graded non-split p-filiform Leibniz algebra $(n - p \geq 4)$ is isomorphic to one of the following non-isomorphic algebras:*

if $p = 2k$, then

$$\mu_1 : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n - 2k - 1, \\ [e_1, f_j] = f_{k+j}, & 1 \leq j \leq k, \end{cases}$$

$$\mu_2 : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n - 2k - 1, \\ [e_1, f_1] = e_2 + f_{k+1}, & \\ [e_i, f_1] = e_{i+1}, & 2 \leq i \leq n - 2k - 1, \\ [e_1, f_j] = f_{k+j}, & 2 \leq j \leq k, \end{cases}$$

if $p = 2k + 1$, then

$$\mu_3 : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n - 2k - 2, \\ [e_1, f_j] = f_{k+1+j}, & 1 \leq j \leq k, \\ [e_i, f_{k+1}] = e_{i+1}, & 1 \leq i \leq n - 2k - 2, \end{cases}$$

where $\{e_1, e_2, \ldots, e_{n-p}, f_1, f_2, \ldots, f_p\}$ is the basis of the algebra and the omitted products are equal to zero.

In order to simplify our further calculations for the algebra μ_3, by taking the change of basis in the following form:

$$e_1' = e_1, \quad e_2' = e_1 - f_{k+1}, \quad e_{i+1}' = e_i, \ 2 \leq i \leq n - 2k - 1, \quad f_j' = f_j,$$

$$f_{k+j}' = f_{k+1+j}, \ 1 \leq j \leq k,$$

we obtain the table of multiplication of the algebra μ_3, which we shall use throughout the paper:

$$\mu_3 : \begin{cases} [e_1, e_1] = e_3, & \\ [e_i, e_1] = e_{i+1}, & 2 \leq i \leq n - 2k - 1, \\ [e_1, f_j] = f_{k+j}, & 1 \leq j \leq k, \\ [e_2, f_j] = f_{k+j}, & 1 \leq j \leq k. \end{cases}$$

3. $\frac{1}{2}$-derivations of naturally graded non-Lie p-filiform Leibniz algebras

In order to start the description we need to know the $\frac{1}{2}$-derivations of naturally graded non-Lie p-filiform Leibniz algebras.

Proposition 3.1. *Any $\frac{1}{2}$-derivations of the algebra μ_1 has the following matrix form:*

$$\begin{aligned}
D(e_1) &= \sum_{j=1}^{n-2k} a_j e_j + \sum_{j=1}^{2k} b_j f_j,\\
D(e_2) &= a_1 e_2 + 2^{-1}\sum_{j=3}^{n-2k} a_{j-1} e_j + 2^{-1}\sum_{j=1}^{k} b_j f_{k+j},\\
D(e_i) &= a_1 e_i + 2^{-i+1}\sum_{j=i+1}^{n-2k} a_{j-i+1} e_j, \quad 3 \le i \le n-2k,\\
D(f_i) &= c_{n-2k,i} e_{n-2k} + \sum_{j=1}^{2k} d_{j,i} f_j, \quad 1 \le i \le k,\\
D(f_{k+i}) &= 2^{-1} a_1 f_{k+i} + 2^{-1}\sum_{j=1}^{k} d_{j,i} f_{k+j}, \quad 1 \le i \le k.
\end{aligned}$$

Proof. Let $\{e_1, f_1, f_2, \dots, f_k\}$ be a generator basis elements of the algebra μ_1.

We put

$$D(e_1) = \sum_{i=1}^{n-2k} a_i e_i + \sum_{i=1}^{2k} b_i f_i, \qquad D(f_i) = \sum_{j=1}^{n-2k} c_{j,i} e_j + \sum_{j=1}^{2k} d_{j,i} f_j, \quad 1 \le i \le k.$$

From the $\frac{1}{2}$-derivation property (2.6) we have

$$\begin{aligned}
D(e_2) = D([e_1,e_1]) &= \frac{1}{2}\left([D(e_1), e_1] + [e_1, D(e_1)]\right) =\\
&= \frac{1}{2}\left([\sum_{i=1}^{n-2k} a_i e_i + \sum_{i=1}^{2k} b_i f_i, e_1] + [e_1, \sum_{i=1}^{n-2k} a_i e_i + \sum_{i=1}^{2k} b_i f_i]\right) =\\
&= a_1 e_2 + 2^{-1}\sum_{j=3}^{n-2k} a_{j-1} e_j + 2^{-1}\sum_{j=1}^{k} b_j f_{k+j}.
\end{aligned}$$

Buy applying the induction and the $\frac{1}{2}$-derivation property (2.6) we derive

$$D(e_i) = a_1 e_i + 2^{-i+1}\sum_{j=i+1}^{n-2k} a_{j-i+1} e_j, \quad 3 \le i \le n-2k.$$

Consider

$$\begin{aligned}0 = D([f_i, e_1]) &= \frac{1}{2}\left([D(f_i), e_1] + [f_i, D(e_1)]\right) = \\ &= \frac{1}{2}\left([\sum_{j=1}^{n-2k} c_{j,i}e_j + \sum_{j=1}^{2k} d_{j,i}f_j, e_1] + [f_i, \sum_{i=1}^{n-2k} a_ie_i + \sum_{i=1}^{2k} b_if_i]\right) = \\ &= 2^{-1}\sum_{j=1}^{n-2k-1} c_{j,i}e_{j+1}, \quad 1 \le i \le k.\end{aligned}$$

Consequently,

$$c_{j,i} = 0, \quad 1 \le i \le k, \quad 1 \le j \le n-2k-1.$$

Similarly, from $D(f_{k+i}) = D([e_1, f_i]),\ 1 \le i \le k$, we deduce

$$D(f_{k+i}) = 2^{-1}a_1f_{k+i} + 2^{-1}\sum_{j=1}^{k} d_{j,i}f_{k+j}, \qquad 1 \le i \le k.$$

Complete the proof of proposition. □

Proposition 3.2. *Any $\frac{1}{2}$-derivation of the algebra μ_2 has the following matrix form:*

$$\begin{aligned}D(e_1) &= \sum_{j=1}^{n-2k} a_je_j + \sum_{j=1}^{2k} b_jf_j, \\ D(e_2) &= (a_1 + 2^{-1}b_1)e_2 + 2^{-1}\sum_{j=3}^{n-2k} a_{j-1}e_j + 2^{-1}\sum_{j=1}^{k} b_jf_{k+j}, \\ D(e_i) &= (a_1 + (1-2^{1-i})b_1)e_i + 2^{1-i}\sum_{j=i+1}^{n-2k} a_{j-i+1}e_j, \quad 3 \le i \le n-2k, \\ D(f_1) &= c_{n-2k,1}e_{n-2k} + (a_1+b_1)f_1 + \sum_{j=2}^{2k} d_{j,1}f_j, \\ D(f_i) &= c_{n-2k,i}e_{n-2k} + \sum_{j=2}^{2k} d_{j,i}f_j, \quad 2 \le i \le k, \\ D(f_{k+1}) &= a_1f_{k+1} + 2^{-1}\sum_{j=1}^{k}(d_{j,1} - b_j)f_{k+j}, \\ D(f_{k+i}) &= 2^{-1}a_1f_{k+i} + 2^{-1}\sum_{j=2}^{k} d_{j,i}f_{k+j}, \quad 2 \le i \le k.\end{aligned}$$

Proof. The proof follows by straightforward calculations similarly to the proof of Proposition 3.1. □

Proposition 3.3. *Any $\frac{1}{2}$-derivation of the algebra μ_3 has the following matrix form:*

$$D(e_1) = \sum_{j=1}^{n-2k} a_j e_j + \sum_{j=1}^{2k} b_j f_j,$$

$$D(e_2) = (a_1 + a_2)e_2 + \sum_{j=3}^{n-2k-1} a_j e_j + \alpha_{n-2k} e_{n-2k} + \sum_{j=k+1}^{2k} \beta_j f_j,$$

$$D(e_3) = (a_1 + 2^{-1}a_2)e_3 + \frac{1}{2}\sum_{j=4}^{n-2k} a_{j-1} e_j + \frac{1}{2}\sum_{j=1}^{k} b_j f_{k+j},$$

$$D(e_i) = (a_1 + 2^{2-i}a_2)e_i + 2^{2-i}\sum_{j=i+1}^{n-2k} a_{j-i+2} e_j, \quad 4 \le i \le n-2k,$$

$$D(f_i) = c_{n-2k,i} e_{n-2k} + \sum_{j=1}^{2k} d_{j,i} f_j, \quad 1 \le i \le k,$$

$$D(f_{k+i}) = 2^{-1}(a_1 + a_2) f_{k+i} + 2^{-1}\sum_{j=1}^{k} d_{j,i} f_{k+j}, \quad 1 \le i \le k.$$

Proof. The proof follows by straightforward calculations similarly to the proof of Proposition 3.1. □

Remark 3.4. *The dimensions of the space of $\frac{1}{2}$-derivation of the algebras μ_1, μ_2 and μ_3 are*

$$\begin{aligned} \dim \mathrm{Der}_{\frac{1}{2}}(\mu_1) &= n + 2k^2 + k, \\ \dim \mathrm{Der}_{\frac{1}{2}}(\mu_2) &= n + 2k^2 + 1, \\ \dim \mathrm{Der}_{\frac{1}{2}}(\mu_3) &= n + 2k^2 + 2k + 1, \end{aligned}$$

where $k \in \mathbb{N}$ and $n \ge 2k+4$.

4. Local and 2-local $\frac{1}{2}$-derivation of naturally graded non-Lie p-filiform Leibniz algebras

Now we shall give the main result concerning local and 2-local $\frac{1}{2}$-derivations of naturally graded non-Lie p-filiform Leibniz algebras.

In the following theorem we give the description of local $\frac{1}{2}$-derivation of the algebra μ_1.

Theorem 4.1. *Let Δ be a linear operator on μ_1. Then Δ is a local $\frac{1}{2}$-derivation, if and only if its matrix* *has the form:*

$$\begin{aligned}
\Delta(e_1) &= \sum_{i=1}^{n-2k}\gamma_{i,1}e_i + \sum_{i=1}^{2k}\gamma_{n-2k+i,1}f_i,\\
\Delta(e_2) &= \sum_{i=2}^{n-2k}\gamma_{i,2}e_i + \sum_{i=k+1}^{2k}\gamma_{n-2k+i,2}f_i,\\
\Delta(e_i) &= \sum_{i=j}^{n-2k}\gamma_{i,j}e_i, \quad 3 \le j \le n-2k, \qquad (1)\\
\Delta(f_j) &= \gamma_{n-2k,n-2k+j}e_{n-2k} + \sum_{i=1}^{2k}\gamma_{n-2k+i,n-2k+j}f_i, \quad 1 \le j \le k,\\
\Delta(f_j) &= \sum_{i=k+1}^{2k}\gamma_{n-2k+i,n-2k+j}f_i, \quad k+1 \le j \le 2k.
\end{aligned}$$

Proof. ($\Rightarrow$) Assume that Δ is a local $\frac{1}{2}$-derivation of μ_1 :

$$\begin{aligned}
\Delta(e_i) &= \sum_{i=1}^{n-2k}\gamma_{i,j}e_i + \sum_{i=1}^{2k}\gamma_{n-2k+i,j}f_i, \quad 1 \le j \le n,\\
\Delta(f_j) &= \sum_{i=1}^{n-2k}\gamma_{i,n-2k+j}e_i + \sum_{i=1}^{2k}\gamma_{n-2k+i,n-2k+j}f_i, \quad 1 \le j \le 2k.
\end{aligned}$$

Take a $\frac{1}{2}$-derivation D_{e_2} such that $\Delta(e_2) = D_{e_2}(e_2)$. Then

$$\begin{aligned}
\Delta(e_2) &= \sum_{i=1}^{n-2k}\gamma_{i,2}e_i + \sum_{i=1}^{2k}\gamma_{n-2k+i,2}f_i,\\
D_{e_2}(e_2) &= a_1e_2 + 2^{-1}\sum_{j=3}^{n-2k}a_{j-1}e_j + 2^{-1}\sum_{j=1}^{k}b_jf_{k+j}.
\end{aligned}$$

Comparing the coefficients, we conclude that $\gamma_{1,2} = \gamma_{n-2k+i,2} = 0$ for $1 \le i \le k$.
We take a $\frac{1}{2}$-derivation D_{e_j} such that $\Delta(e_j) = D_{e_j}(e_j)$, where $3 \le j \le n-2k$. Then

$$\begin{aligned}
\Delta(e_j) &= \sum_{i=1}^{n-2k}\gamma_{i,j}e_i + \sum_{i=1}^{2k}\gamma_{n-2k+i,j}f_i,\\
D_{e_j}(e_j) &= a_1e_j + 2^{-j+1}\sum_{i=j+1}^{n-2k}a_{i-j+1}e_i.
\end{aligned}$$

Comparing the coefficients at the basis elements for $\Delta(e_j)$ and $D_{e_j}(e_j)$, we obtain the identities

$$\gamma_{t,j} = \gamma_{n-2k+i,j} = 0, \quad 3 \le j \le n-2k,\ 1 \le i \le 2k,\ 1 \le t \le j-1.$$

We take a $\frac{1}{2}$-derivation D_{f_j} such that $\Delta(f_j) = D_{f_j}(f_j)$, where $1 \le j \le k$. Then

$$\begin{aligned} \Delta(f_j) = & \sum_{i=1}^{n-2k} \gamma_{i,n-2k+j} e_i + \sum_{i=1}^{2k} \gamma_{n-2k+i,n-2k+j} f_i, \\ D_{f_j}(f_j) = & \ c_{n-2k,j} e_{n-2k} + \sum_{i=1}^{2k} d_{i,j} f_i. \end{aligned}$$

Comparing the coefficients at the basis elements for $\Delta(f_j)$ and $D_{f_j}(f_j)$, we obtain

$$\gamma_{i,n-2k+j} = 0, \quad 1 \le j \le k, \ 1 \le i \le n-2k-1.$$

Now, take a $\frac{1}{2}$-derivation D_{f_j} such that $\Delta(f_j) = D_{f_j}(f_j)$, where $k+1 \le j \le 2k$. Then

$$\begin{aligned} \Delta(f_j) = & \sum_{i=1}^{n-2k} \gamma_{i,n-2k+j} e_i + \sum_{i=1}^{2k} \gamma_{n-2k+i,n-2k+j} f_i, \\ D_{f_j}(f_j) = & \ 2^{-1} a_1 f_{k+i} + 2^{-1} \sum_{j=1}^{k} d_{j,i} f_{k+j}, \end{aligned}$$

which implies

$$\gamma_{i,n-2k+j} = 0, \quad k+1 \le j \le 2k, \ 1 \le i \le n-k.$$

($\Leftarrow$) Assume that the operator Δ has the form (1). For an arbitrary element

$$x = \sum_{i=1}^{n-2k} \xi_i e_i + \sum_{i=1}^{2k} \zeta_i f_i,$$

we have

$$\begin{aligned} D(x)_{e_1} = & \ a_1 \xi_1, \\ D(x)_{e_i} = & \ a_i \xi_1 + \sum_{j=1}^{i-2} 2_1^{-j} a_{i-j} \xi_{j+1} + a_1 \xi_i, \quad 2 \le i \le n-2k-1, \\ D(x)_{e_{n-2k}} = & \ a_{n-2k} \xi_1 + \sum_{j=1}^{n-2k-2} 2^{-j} a_{n-2k-j} \xi_{j+1} + a_1 \xi_{n-2k} + \sum_{j=1}^{k} c_{n-2k,j} \zeta_j, \\ D(x)_{f_i} = & \ b_i \xi_1 + \sum_{j=1}^{k} d_{i,j} \zeta_j, \qquad 1 \le i \le k, \\ D(x)_{f_i} = & \ b_i \xi_1 + 2^{-1} b_{i-k} \xi_2 + \sum_{j=1}^{k} d_{i,j} \zeta_j + \\ & +2^{-1} a_1 \zeta_i + 2^{-1} \sum_{j=1}^{k} d_{i-k,j} \zeta_{k+j}, \quad k+1 \le i \le 2k. \end{aligned}$$

The coordinates of $\Delta(x)$ are

$$\begin{aligned}
\Delta(x)_{e_1} &= \gamma_{11}\xi_1,\\
\Delta(x)_{e_i} &= \sum_{j=1}^{i}\gamma_{i,j}\xi_j, \quad 2\le i\le n-2k-1,\\
\Delta(x)_{e_{n-2k}} &= \sum_{j=1}^{n-2k}\gamma_{n-2k,j}\xi_j+\sum_{j=1}^{k}\gamma_{n-2k,n-2k+j}\zeta_j,\\
\Delta(x)_{f_i} &= \gamma_{n-2k+i,1}\xi_1+\sum_{j=1}^{k}\gamma_{n-2k+i,n-2k+j}\zeta_j, \qquad 1\le i\le k,\\
\Delta(x)_{f_i} &= \gamma_{n-2k+i,1}\xi_1+\gamma_{n-2k+i,2}\xi_2+\sum_{j=1}^{2k}\gamma_{n-2k+i,n-2k+j}\zeta_j,\ k+1\le i\le 2k.
\end{aligned}$$

Comparing the coordinates of $\Delta(x)$ and $D(x)$, we obtain

$$\begin{cases}
a_1\xi_1 & =\gamma_{11}\xi_1\\
a_i\xi_1+\sum_{j=1}^{i-2}2_1^{-j}a_{i-j}\xi_{j+1}+a_1\xi_i & =\sum\limits_{j=1}^{i}\gamma_{i,j}\xi_j,\quad 2\le i\le n-2k-1,\\
a_{n-2k}\xi_1+\sum_{j=1}^{n-2k-2}2^{-j}a_{n-2k-j}\xi_{j+1}+ & \\
+a_1\xi_{n-2k}+\sum_{j=1}^{k}c_{n-2k,j}\zeta_j & =\sum\limits_{j=1}^{n-2k}\gamma_{n-2k,j}\xi_j+\sum\limits_{j=1}^{k}\gamma_{n-2k,n-2k+j}\zeta_j,\\
b_i\xi_1+\sum_{j=1}^{k}d_{i,j}\zeta_j & =\gamma_{n-2k+i,1}\xi_1+\\
 & +\sum\limits_{j=1}^{k}\gamma_{n-2k+i,n-2k+j}\zeta_j,\quad 1\le i\le k,\\
b_i\xi_1+2^{-1}b_{i-k}\xi_2+\sum_{j=1}^{k}d_{i,j}\zeta_j+ & \\
+2^{-1}a_1\zeta_i+2^{-1}\sum_{j=1}^{k}d_{i-k,j}\zeta_{k+j} & =\gamma_{n-2k+i,1}\xi_1+\gamma_{n-2k+i,2}\xi_2+\\
 & +\sum\limits_{j=1}^{2k}\gamma_{n-2k+i,n-2k+j}\zeta_j,\quad k+1\le i\le 2k.
\end{cases} \tag{2}$$

We show the solvability of this system of equations with respect to a_i, b_i, c_i and $d_{i,j}$. For this purpose we consider the following possible cases.

Case 1. Let $\xi_1\neq 0$, then putting $c_i=d_{i,j}=0,\ 1\le i\le k,\ 1\le j\le k$ from (2) we uniquely determine $a_1,a_2,\dots,a_{n-2k},b_1,b_2,\dots,b_{2k}$.

Case 2. Let $\xi_1=0$ and $\xi_2\neq 0$, then putting $a_1=c_i=d_{i,j}=0,\ 1\le i\le k,\ 1\le j\le k$ we uniquely determine remaining unknowns $a_2,\dots,a_{n-2k},b_1,b_2,\dots,b_k$.

Case 3. Let $\xi_1=\xi_2=\dots=\xi_{r-1}=0$ and $\xi_r\neq 0,\ 3\le r\le n-2k$. Then putting

$$a_1=\ldots=a_{n-2k-m}=b_1=\ldots=c_1=\ldots=d_{t,j}=0,\ 1\le t\le k,\ 1\le j\le k.$$

we determine unknowns $a_{n-2k-m+1},\ i\le m\le n-2k$.

Case 4. Let $\xi_1=\ldots=\xi_{n-2k}=\zeta_1=\ldots=\zeta_{r-1}=0$ and $\zeta_r\neq 0,\ 1\le r\le k$. Then setting

$$a_1=\ldots=b_1=\ldots=0,\ c_i=0,\ i\neq r,\ d_{j,i}=0,\ j\neq r,$$

we determine $c_r,d_{r,i},\ 1\le i\le k$.

Case 5. Let $\xi_1=\ldots=\xi_{n-2k}=\zeta_1=\ldots=\zeta_{k+r-1}=0$ and $\zeta_{k+r}\neq 0,\ 1\le r\le k$. Then setting

$$a_1=\ldots=b_1=\ldots=c_1=\ldots=0,\ d_{j,i}=0,\ r\neq k+r,$$

we obtain that the unknowns $d_{k+r,i}, k+1\le i\le 2k$, are uniquely determined from (2). □

In the following theorems we obtain the descriptions of local $\frac{1}{2}$-derivation of the algebras μ_2 and μ_3.

Theorem 4.2. *Let Δ be a linear operator on μ_2. Then Δ is a local $\frac{1}{2}$-derivation, if and only if its matrix has the form:*

$$\begin{aligned}
\Delta(e_1) &= \sum_{i=1}^{n-2k} \gamma_{i,1} e_i + \sum_{i=1}^{2k} \gamma_{n-2k+i,1} f_i,\\
\Delta(e_2) &= \sum_{i=2}^{n-2k} \gamma_{i,2} e_i + \sum_{i=k+1}^{2k} \gamma_{n-2k+i,2} f_i,\\
\Delta(e_i) &= \sum_{i=j}^{n-2k} \gamma_{i,j} e_i, \quad 3 \le j \le n-2k,\\
\Delta(f_1) &= \gamma_{n-2k,n-2k+1} e_{n-2k} + \sum_{i=1}^{2k} \gamma_{n-2k+i,n-2k+1} f_i,\\
\Delta(f_j) &= \gamma_{n-2k,n-2k+j} e_{n-2k} + \sum_{i=2}^{2k} \gamma_{n-2k+i,n-2k+j} f_i, \quad 2 \le j \le k,\\
\Delta(f_{k+1}) &= \sum_{i=k+1}^{2k} \gamma_{n-2k+i,n-k+1} f_i,\\
\Delta(f_j) &= \sum_{i=k+2}^{2k} \gamma_{n-2k+i,n-2k+j} f_i, \quad k+2 \le j \le 2k.
\end{aligned}$$

Proof. The proof is similar to the proof of Theorem 4.1 □

Theorem 4.3. *Let Δ be a linear operator on μ_3. Then Δ is a local $\frac{1}{2}$-derivation if and only if its matrix has the form:*

$$\begin{aligned}
\Delta(e_1) &= \sum_{i=1}^{n-2k} \gamma_{i,1} e_i + \sum_{i=1}^{2k} \gamma_{n-2k+i,1} f_i,\\
\Delta(e_2) &= \gamma_{2,2} e_2 + \sum_{i=3}^{n-2k-1} \gamma_{i,1} e_i + \gamma_{n-2k,2} e_{n-2k} + \sum_{i=k+1}^{2k} \gamma_{n-2k+i,2} f_i,\\
\Delta(e_i) &= \sum_{i=j}^{n-2k} \gamma_{i,j} e_i, \quad 3 \le j \le n-2k,\\
\Delta(f_j) &= \gamma_{n-2k,n-2k+j} e_{n-2k} + \sum_{i=1}^{2k} \gamma_{n-2k+i,n-2k+j} f_i, \quad 1 \le j \le k,\\
\Delta(f_j) &= \sum_{i=k+1}^{2k} \gamma_{n-2k+i,n-2k+j} f_i, \quad k+1 \le j \le 2k.
\end{aligned}$$

Proof. The proof is similar to the proof of Theorem 4.1 □

Example 4.4. *Let μ_1 naturally graded non-Lie p-filiform Leibniz algebras, admits a local $\frac{1}{2}$-derivation which is not a $\frac{1}{2}$-derivation.*

Proof. Let us consider the operator Δ, such that

$$\Delta\left(\sum_{i=1}^{n-2k} x_i e_i + \sum_{i=1}^{2k} y_i f_i\right) = 3x_1 e_{n-2k-1} + \frac{1}{2} x_2 e_{n-2k}.$$

By Proposition 3.1, it is not difficult to see that Δ is not a $\frac{1}{2}$-derivation. We show that Δ is a local $\frac{1}{2}$-derivation. Consider the $\frac{1}{2}$-derivations D_1 and D_2 of the algebra μ_1 are given of the forms:

$$D_1\left(\sum_{i=1}^{n-2k} x_i e_i + \sum_{i=1}^{2k} y_i f_i\right) = x_1 e_{n-2k-1} + \frac{1}{2} x_2 e_{n-2k},$$
$$D_2\left(\sum_{i=1}^{n-2k} x_i e_i + \sum_{i=1}^{2k} y_i f_i\right) = x_1 e_{n-2k}.$$

We prove that for any $x = \sum_{i=1}^{n-2k} x_i e_i + \sum_{i=1}^{2k} y_i f_i$ there exist a $\frac{1}{2}$-derivation D such that $\Delta(x) = D(x)$.

If $x_1 = 0$, then

$$\Delta(x) = \frac{1}{2} x_2 e_{n-2k} = D_1(x).$$

If $x_1 \neq 0$, then setting $D = D_1 + tD_2$, where $t = -\frac{x_2}{x_1}$, we get

$$D(x) = 3D_1(x) + tD_2(x) = 3\left(x_1 e_{n-2k-1} + \frac{1}{2} x_2 e_{n-2k}\right) + t x_1 e_{n-2k} = 3x_1 e_{n-2k-1} +$$
$$+ \frac{3}{2} x_2 e_{n-2k} - x_2 e_{n-2k} = 3x_2 e_{n-2k-1} + \frac{1}{2} x_2 e_{n-2k} = \Delta(x).$$

Hence, Δ is a local $\frac{1}{2}$-derivation. □

Remark 4.5. *The dimensions of the space of local $\frac{1}{2}$-derivation of algebras μ_1, μ_2 and μ_3 are*

$$\dim \mathrm{LocDer}_{\frac{1}{2}}(\mu_1) = \frac{n^2 + 10k^2 - 4kn + n + 6k}{2},$$
$$\dim \mathrm{LocDer}_{\frac{1}{2}}(\mu_2) = \frac{n^2 + 10k^2 - 4kn + n + 2k + 4}{2},$$
$$\dim \mathrm{LocDer}_{\frac{1}{2}}(\mu_3) = \frac{n^2 + 10k^2 - 4kn - n + 12k + 4}{2},$$

where $k \in \mathbb{N}$ and $n \geq 2k + 4$.

Remarks 3.4 and 4.5 show that the dimensions of the spaces of all local $\frac{1}{2}$-derivation of the algebras μ_i, $i = 1, 2, 3$, are strictly greater than the dimensions of the space of all $\frac{1}{2}$-derivation of μ_i. Therefore, we have the following result.

Corollary 4.6. *The algebras μ_1, μ_2 and μ_3 admit local $\frac{1}{2}$-derivation which are not $\frac{1}{2}$-derivation.*

Now we examine 2-local $\frac{1}{2}$-derivation of p-filiform Leibniz algebras.

Theorem 4.7. *The algebras μ_1, μ_2 and μ_3 admit 2-local $\frac{1}{2}$-derivations which are not $\frac{1}{2}$-derivations.*

Proof. We prove the theorem for the algebra μ_1; for μ_2 and μ_3, the proof are similar. Consider a homogeneous non-additive function on $\mathbb{C}^2$, such that

$$f(z_1, z_2) = \begin{cases} \frac{z_1^2}{z_2}, & \text{if } z_2 \neq 0, \\ 0, & \text{if } z_2 = 0. \end{cases}$$

Consider the mapping $\nabla : \mu_1 \to \mu_1$ defined by the rule

$$\nabla(x) = \ f(\xi_1, \zeta_1)e_{n-2k}, \quad x = \sum_{i=1}^{n-2k} \xi_i e_i + \sum_{i=1}^{2k} \zeta_i f_i \in \mu_1.$$

Since f is non-additive, we obtain that ∇ is not a $\frac{1}{2}$-derivation.

Let us show that ∇ is a 2-local $\frac{1}{2}$-derivation. We consider arbitrary elements

$$x = \sum_{i=1}^{n-2k} \xi_i^{(x)} e_i + \sum_{i=1}^{2k} \zeta_i^{(x)} f_i, \quad y = \sum_{i=1}^{n-2k} \xi_i^{(y)} e_i + \sum_{i=1}^{2k} \zeta_i^{(y)} f_i.$$

We search a derivation D of the form:

$$D \ = \begin{pmatrix} D_{1,1} & D_{1,2} \\ 0 & 0 \end{pmatrix},$$

where $D_{1,1} = a_{n-2k} e_{n-2k,1}$ and $D_{1,2} = c_1 e_{n-2k,n-2k+1}$,

such that that $\nabla(x) = D(x)$ and $\nabla(y) = D(y)$. From this equality we obtain the following system of equations for a_{n-2k} and c_1:

$$\begin{cases} \xi_1^{(x)} a_{n-2k} + \zeta_1^{(x)} c_1 = f\left(\xi_1^{(x)}, \zeta_1^{(x)}\right), \\ \xi_1^{(y)} a_{n-2k} + \zeta_1^{(y)} c_1 = f\left(\xi_1^{(y)}, \zeta_1^{(y)}\right). \end{cases} \tag{3}$$

Case 1. Let $\xi_1^{(x)}\zeta_1^{(y)} - \xi_1^{(y)}\zeta_1^{(x)} = 0$, then, the system (3) has infinitely many solutions, because of the right-hand side is homogeneous.

Case 2. Let $\xi_1^{(x)}\zeta_1^{(y)} - \xi_1^{(y)}\zeta_1^{(x)} \neq 0$. In this case, the system (3) has a unique solution.

□

Acknowledgment: I am very grateful to the referee for the careful study of our article and significant comments.

References

[1] K. K. Abdurasulov, A. Sh. Ayupov and B. B. Yusupov, Local and 2-local derivations on filiform associative algebras, Journal of Algebra its Applications, 24(10) (2025) 2550242.

[2] J. Q. Adashev, B. B. Yusupov, Local derivations and automorphisms of direct sum null-filiform Leibniz algebras, Lobachevskii Journal of Mathematics 43(12) (2022) 1–7.

[3] A. K. Alauadinov, B. B. Yusupov, Local derivations of conformal Galilei algebra, Communications in Algebra 52(6) (2024) 2489–2508.

[4] Sh. Ayupov, A. Elduque and K. Kudaybergenov, Local derivations and automorphisms of Cayley algebras, Journal of Pure and Applied Algebra, 227(5) (2023) 107277.
[5] Sh. Ayupov, K. Kudaybergenov, I. Rakhimov, 2-Local derivations on finite-dimensional Lie algebras, Linear Algebra and its Applications, 474 (2015) 1–11.
[6] Sh. Ayupov, K. Kudaybergenov, Local derivations on finite-dimensional Lie algebras, Linear Algebra and its Applications, 493 (2016) 381–398.
[7] Sh. A. Ayupov, K. K. Kudaybergenov, B. B. Yusupov, Local and 2-local derivations of p-filiform Leibniz algebras, Journal of Mathematical Sciences, 245(3) (2020) 359–367.
[8] Sh. A. Ayupov, K. K. Kudaybergenov, B. B. Yusupov, Local and 2-Local Derivations of Locally Simple Lie Algebras, Journal of Mathematical Sciences, 278(4) (2024) 613–622.
[9] Sh. Ayupov, K. Kudaybergenov, B. Yusupov, 2-Local derivations on generalized Witt algebras, Linear and Multilinear Algebra, 69(16) (2021) 3130–3140.
[10] Sh. Ayupov, A. Khudoyberdiyev, Local derivations on solvable Lie algebras, Linear and Multilinear Algebra, 69(7) (2021) 1286–1301.
[11] Sh. Ayupov, A. Khudoyberdiyev, B. Yusupov, Local and 2-local derivations of solvable Leibniz algebras, International Journal of Algebra and Computation, 30(6) (2020) 1185–1197.
[12] Sh. Ayupov, B. Yusupov, 2-local derivations of infinite-dimensional Lie algebras, Journal of Algebra and its Applications, 19:5 (2020) ID 2050100.
[13] Z. Chen, D. Wang, 2-Local automorphisms of finite-dimensional simple Lie algebras, Linear Algebra and its Applications, 486 (2015) 335–344.
[14] Y. Chen, K. Zhao, Y. Zhao, Local derivations on Witt algebras, Linear and Multilinear algebra, 70(6) (2022) 1159–1172.
[15] L. M. Camacho, J. R. Gomez , A. J. Gonzalez and B. A. Omirov, The classification of naturally graded p-filiform Leibniz algebras, Communications in Algebra, 39(1) (2010) 153–168.
[16] V.Filippov, On δ-derivations of Lie algebras, Siberian Mathematical Journal, 39(6) (1998) 1218–1230.
[17] V.Filippov, δ-derivations of prime Lie algebras, Siberian Mathematical Journal, 40(6) (1999) 174–184.
[18] V.Filippov, On δ-derivations of prime alternative and Malcev algebras, Algebra and Logic, 39(5) (2000) 354–358.
[19] Q. Jiang, X. Tang, 2-Local derivations on the Schrödinger-Virasoro algebra, Linear and Multilinear Algebra, 72(8) (2023) 1328–1345.
[20] R.V. Kadison, Local derivations, Journal of Algebra, 130 (1990) 494–509.
[21] K. Kudaybergenov, B. Omirov, T. Kurbanbaev, Local derivations on solvable Lie algebras of maximal rank, Communications in Algebra, 50(9) (2022) 1–11.
[22] A. Kh. Khudoyberdiyev, B. B. Yusupov, Local and 2-local $\frac{1}{2}$-derivation on finite-dimensional Lie algebras, Results in Mathematics, 79 (2024) 210.
[23] U. Kh. Mamadaliyev, A. Sattarov, B. B. Yusupov, Local and 2-local $\frac{1}{2}$-derivations of solvable Leibniz algebras, Eurasian Mathematical Journal, 16 (2) (2025) 42–54.
[24] D. R. Larson, A. R. Sourour, Local derivations and local automorphisms of $B(X)$, Proceedings of Symposia in Pure Mathematics, 51 (1990) 187–194.
[25] P. Šemrl, Local automorphisms and derivations on $B(H)$, Proceedings of the American Mathematical Society, 125 (1997) 2677–2680.
[26] X. Tang, M. Xiao and P. Wang, Local properties of Virasoro-like algebra, Journal of Geometry and Physics, 186 (2023) 104772.
[27] Y. F. Yao, Local derivations on the Witt algebra in prime characteristic, Linear and multilinear algebra, 70(15) (2022) 2919–2933.
[28] Y. Yao, K. Zhao, Local properties of Jacobson-Witt algebras, Journal of Algebra, 586 (2021) 1110–1121.
[29] B. B. Yusupov, N. Z. Vaisova and T. Madrakhimov, Local anti-derivation on of n-dimensional naturally graded quasi-filiform Leibniz algebra of type I, AIP Conference Proceedings, 3147 (1) (2024) 020008.