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On strongly semicommutative modules*

Research Article

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Abstract: For a left module ${}_RM$ over a non-commutative ring R, we define the concept of a strongly semicommutative module as a generalization of the reduced module. This notion constitutes a distinct and stronger category within the class of semicommutative modules. We demonstrate that a module ${}_RM$ is strongly semicommutative if and only if ${}_{A_n(R)}A_n(M)$ is strongly semicommutative. Additionally, we establish that ${}_RM$ is strongly semicommutative if and only if ${}_{R[x]}M[x]$ is strongly semicommutative; this is also equivalent to ${}_{R[x,x^{-1}]}M[x,x^{-1}]$ being strongly semicommutative. Among our findings, we prove that if ${}_RM$ is strongly semicommutative, then for any reduced submodule N of M, the quotient module M/N is also strongly semicommutative. We provide examples of semicommutative modules that are not strongly semicommutative and show that the class of strongly semicommutative modules remains closed under localization.

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1. Introduction

In this article, R represents a ring with identity, and ${}_RM$ represents a unital left R-module. Recall that for some $n \in \mathbb{N}$ and $a \in R$, if $a^n = 0$, then a is said to be a nilpotent element in R. The notation $Nil(R)$ denotes the set of all nilpotent elements in R. If $Nil(R) = \{0\}$, R is called a reduced ring. R is called zero-insertive (semicommutative) if the condition $uRv = 0$ holds true whenever $u, v \in R$ satisfy $uv = 0$. The study of semicommutative rings and their various generalizations was carried out by Kim and Lee in [9], Gang in [6], Huh et al. in [8], Yang in [15], and Liu in [12]. In [5], Cohn introduced a new concept called a reversible ring as a stronger condition within the class of semicommutative rings. R is called reversible if $uv = 0$ implies $vu = 0$ for all $u, v \in R$. Thus, we have the following implications:

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reduced (also, commutative) $\Rightarrow$ reversible $\Rightarrow$ semicommutative.

The polynomial extension of the semicommutative property has been of keen interest to many algebraists. In this regard, Hirano claimed that if R is semicommutative, then $R[x]$ is also semicommutative. However, Example 2 in [8] disproved this possibility. Nielsen in [13] constructed an example of a semicommutative ring, which is not McCoy. A ring R is said to be right McCoy(left McCoy) if for each pair of non-zero polynomial $p(x)$, $q(x) \in R[x]$ with $p(x)q(x) = 0$ then there exists a non-zero element $r \in R$ with $p(x)r = 0$(respectively $rq(x) = 0$). A ring is McCoy if it is both left and right McCoy. Motivated by these results, Gang and Ruijuan in [7] introduced the concept of a strongly semicommutative ring as a new and independent category within the class of semicommutative rings. A ring R is called strongly semicommutative if the condition $p(x)R[x]q(x) = 0$ holds true whenever $p(x) = \sum_{l=0}^{n} u_l x^l$ and $q(x) = \sum_{k=0}^{t} v_k x^k$ in $R[x]$ satisfy $p(x)q(x) = 0$. Lee and Zhou in [11] extended the reduced property to modules in extensions. A module $_RM$ is reduced if it satisfies one of the following equivalent conditions:

(1) If $u^2v = 0$ for some $u \in R$ and $v \in M$, then $uRv = 0$.

(2) If $uv = 0$ for some $u \in R$ and $v \in M$, then $uM \cap Rv = 0$.

Similarly $_RM$ is called rigid if $uv = 0$ holds true whenever $u^2v = 0$ for $u \in R$ and $v \in M$. A module $_RM$ is called Armendariz if $u_l v_k = 0$ whenever $p(x) = \sum_{l=0}^{n} u_l x^l \in R[x]$ and $m(x) = \sum_{k=0}^{q} v_k x^k \in M[x]$ satisfy $p(x).m(x) = 0$. Lee and Zhou recorded many examples of Armendariz modules [12], as well as Rege and Buhphang [4]. They also conducted a comparative study on Armendariz, reduced, and semicommutative modules. A module $_RM$ is semicommutative if, for any $u \in R$ and $v \in M$ that satisfy $uv = 0$, it follows that $uRv = 0$. R is semicommutative if and only if $_RR$ is a semicommutative module. In [3], Buhphang et.al showed that rigid+ semicommutative implies the Armendariz property in modules. However, semicommutative along does not imply Armendariz property. Sufficient examples of modules that are Armendariz but not semicommutative are available in literature. Ansari and Singh in[1], Baser and Agayev[2], and Zhang and Chen in [16] investigated the properties of semicommutative modules and their relation with other classes of modules carried out in Zhang and Chen, in [16], studied the polynomial extension of the semicommutative property and proved that if $_RM$ is Aremdariz as well as semicommutative, then $_{R[x]}M[x]$ semicommutative.

Many researchers have extensively studied the generalization of reduced rings, including Armendariz and semi-commutative rings. However, the lack of definitions for various subclasses has hindered advancements in these areas from extending to modules. In this article, we introduce a new concept known as strongly semicommutative modules, which represents a distinct category within the semicommutative module class. This concept aims to generalize reduced modules within the context of polynomial modules. We examine various properties of this extension and conduct a comparative study between concepts developed in rings and their counterparts in modules. Among our significant results, we demonstrate that a module $_RM$ is strongly semicommutative if and only if $_{A_n(R)}A_n(M)$ is strongly semicommutative. Additionally, we establish that $_RM$ is strongly semicommutative if and only if $_{R[x]}M[x]$ is strongly semicommutative; this condition is also equivalent to $_{R[x,x^{-1}]}M[x, x^{-1}]$ being strongly semicommutative. We prove that if $_RM$ is strongly semicommutative, then for any reduced submodule N of M, the quotient module $\dfrac{M}{N}$ is also strongly semicommutative. Furthermore, we provide examples of semicommutative modules that are not strongly semicommutative and demonstrate that the class of strongly semicommutative modules remains closed under localization.

2. Strongly semicommutative modules

We begin with the following definition.

Definition 2.1. *A module $_RM$ is called strongly semicommutative if the condition $p(x)R[x]m(x) = 0$ holds whenever $p(x) = \sum_{l=0}^{n} u_l x^l \in R[x]$ and $m(x) = \sum_{k=0}^{q} v_k x^k \in M[x]$ satisfy $p(x)m(x) = 0$.*

Lemma 2.2. *Let ${}_RM$ be a module with $p(x) \in R[x]$ and $m(x) \in M[x]$. Then $p(x)R[x]m(x) = 0$ if and only if $p(x)Rm(x) = 0$.*

Proof. Assume $p(x) = \sum_{l=0}^{n} u_l x^l \in R[x]$ and $m(x) = \sum_{k=0}^{q} v_k x^k \in M[x]$ satisfy $p(x)Rm(x) = 0$. Then, for any $c(x) = \sum_{t=0}^{s} w_t x^t \in R[x]$, we have $p(x)c(x)m(x) = p(x)(\sum_{t=0}^{s} w_t x^t)m(x) = \sum_{t=0}^{s} p(x)w_t m(x)x^t = 0$. Thus, $p(x)R[x]m(x) = 0$. The only if part is clear. □

Based on Definition 2.1, we conclude that the class of strongly semicommutative module is closed under submodules and that every reduced module is strongly semicommutative. Moreover, we find that all strongly semicommutative module is semicommutative. However, Example 2.3 given below, demonstrate that the converse does not hold. Moreover, a ring R is strongly semicommutative if and only if the ${}_RR$ is an strongly semicommutative module, and a module ${}_RM$ is strongly semicommutative if and only if ${}_{R[x]}M[x]$ is semicommutative.

Example 2.3. *([7], Example 2.2) Consider a free algebra $M = \mathbb{Z}_2 < d_0, d_1, d_2, d_3, e_0, e_1 >$ generated by six indertiminates over $\mathbb{Z}_2$. Let K be a submodule generated by the following associations:*

$$d_0e_0, d_0e_1 + d_1e_0, d_1e_1 + d_2e_0,$$
$$d_2e_1 + d_3e_0, d_3e_1, d_0e_k(0 \leq k \leq 3), d_3d_k(0 \leq k \leq 3),$$
$$d_1d_k + d_2d_k(0 \leq k \leq 3), e_le_k(0 \leq l, k \leq 1), e_ld_k(0 \leq l \leq 1, 0 \leq k \leq 3).$$

Let $A = M/K$. Neilsen in [13], proved that A is semicommutative but not Mcoy. Further, M is not strongly semicommutative module as consider $p(x) = d_0 + d_1x + d_2x^2 + d_3x^3$ and $m(x) = e_0 + e_1x$. The relations in K suggest that $p(x)m(x) = 0$ in $M[x]$, but however we can see that $p(x)d_0m(x) \neq 0$ since $d_1d_0e_1 + d_0d_1e_2 \notin K$.

Next we noted down a sufficient condition for semicommutative module to implies strongly semicommutative.

Proposition 2.4. *Let ${}_RM$ be an semicommutative module. If ${}_RM$ is Armendariz, then ${}_RM$ is strongly semicommutative module.*

Proof. Let us consider $p(x)m(x) = 0$, where $p(x) = \sum_{l=0}^{n} u_l x^l \in R[x]$ and $m(x) = \sum_{k=0}^{q} v_k x^k \in M[x]$. Since ${}_RM$ being Armendariz implies $u_lv_k = 0$ for each l, k, and since ${}_RM$ being semicommutative implies $u_lrv_k = 0$ for each l, k, it is easy to verify that $p(x)Rm(x) = 0$. Hence, by Lemma 2.2, ${}_RM$ is a strongly semicommutative module. □

Proposition 2.5. *The class of strongly semicommutative modules is closed under direct sums, and direct products.*

Recall that if ${}_RM$ is a submodule of a direct product of copies of R, then ${}_RM$ is considered to be torsionless. Furthermore, R is a submodule of a direct product of copies of a faithful module ${}_RM$. Therefore, Corollary 2.6 and Proposition 2.7 directly follow from Proposition 2.5.

Corollary 2.6. *The following conditions are equivalent for a module ${}_RM$.*

(1) The ring R is strongly semicommutative.

(2) Every submodule of a free module over R is strongly semicommutative.

(3) Every torsionless module over R is strongly semicommutative.

(4) There exists a module ${}_RN$ that is faithful and therefore strongly semicommutative.

Proposition 2.7. *A module ${}_RM$ is strongly semicommutative module if and only if every finitely generated(cyclic) submodules of M is strongly semicommutative.*

For a ring R, we denote $T_n(R)$ as the ring of $n \times n$ upper triangular matrices over R. For a left R-module ${}_RM$ and $K = (a_{ij}) \in M_n(R)$, let $KM = \{(a_{ij}m) : m \in M\}$. For elementary matrices E_{ij}, let $V = \sum_{i=0}^{n} E_{i,i+1}$ for $n \geq 2$. We consider $V_n(R) = RI_n + RV + RV^2 + \cdots + RV^{n-1}$ and $V_n(M) = I_nM + VM + V^2M + \cdots + V^{n-1}M$. Thus,

$$V_n(R) = \left\{ \begin{pmatrix} u_1 & u_2 & \dots & u_{n-1} & u_n \\ 0 & u_1 & u_2 & \dots & u_{n-1} \\ 0 & 0 & u_1 & \dots & u_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & u_1 \end{pmatrix} : u_i \in R,\ 1 \leq i \leq n \right\}.$$

Then, $V_n(R)$ forms a ring, and $V_n(M)$ forms a left module over $V_n(R)$. There exists a ring isomorphism $\phi : V_n(R) \to \frac{R[x]}{(x^n)}$ defined as $\phi(r_0I_n + r_1V + r_2V^2 + \cdots + r_{n-1}V^{n-1}) = r_0 + r_1x + \cdots + r_{n-1}x^{n-1} + (x^n)$, and an abelian group isomorphism $\theta : V_n(M) \to \dfrac{M[x]}{(M[x](x^n))}$ defined as $\theta(m_0I_n + m_1V + m_2V^2 + \cdots + m_{n-1}V^{n-1}) = m_0 + m_1x + \cdots + m_{n-1}x^{n-1} + M[x](x^n)$, such that $\theta(AW) = \phi(A)\theta(W)$ for all $A \in V_n(R)$ and $W \in V_n(M)$. In [9], Kim and Lee proved that if R is reduced, then $T_3(R)$ is a semicommutative ring, whereas $T_n(R)$ is not necessarily semicommutative for $n \geq 4$(see Example 1.3 in [9]). Extending these results for module, Zhang and Chen in [16] proved that if ${}_RM$ is reduced, then $V_n(M)$ is semicommutative over $V_n(R)$ for every $n \in \mathbb{N}$.

By looking into the results related to examples for semicommutative rings and modules, it will interesting to look for some larger submodule of $T_n(M)$ which satisfy the conditions for semicommutative module. Nextly we have generated some extension for the results obtained in [16]. We recall the following notations from [10].

Let $k \in \mathbb{N}$, and for $n = 2k \geq 2$, consider

$$A_n^e(M) = \sum_{i=1}^{k} \sum_{j=k+i}^{n} E_{i,j}M,$$

and for $n = 2k+1 \geq 3$

$$A_n^o(M) = \sum_{i=1}^{k+1} \sum_{j=k+i}^{n} E_{i,j}M.$$

Let

$$A_n(M) = I_nM + VM + \cdots + V^{k-1} + A_n^e(M) \,\text{for}\, n = 2k \geq 2$$

and

$$A_n(M) = I_nM + VM + \cdots + V^{k-1} + A_n^o(M) \,\text{for}\, n = 2k+1 \geq 3.$$

For example,

$$A_4(M) = \left\{ \begin{pmatrix} v_1 & v_2 & v & w \\ 0 & v_1 & v_2 & z \\ 0 & 0 & v_1 & v_2 \\ 0 & 0 & 0 & v_1 \end{pmatrix} : v_1, v_2, v, w, z \in M \right\}.$$

$$A_5(M) = \left\{ \begin{pmatrix} a_1 & a_2 & a & b & c \\ 0 & a_1 & a_2 & d & e \\ 0 & 0 & a_1 & a_2 & f \\ 0 & 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & a_1 \end{pmatrix} : a_1, a_2, a, b, c, d, e, f \in M \right\}.$$

For $A = (a_{ij})$, $B = (b_{ij})$, we write $[A.B]_{ij} = 0$ to mean that $a_{il}b_{lj} = 0$ for $l = 0, \ldots, n$.

Theorem 2.8. *Let $_RM$ be a reduced module. For $n = 2k+1 \geq 3$, the module $A_n(M)$ is a semicommutative over $A_n(R)$*

Proof. Let $a = [a_{ij}] \in A_n(R)$ and $m = [m_{ij}] \in A_n(M)$ satisfy $am = 0$. Firstly, notice that $a = [a_{ij}]$ and $m = [m_{ij}]$ have following properties:

$$\begin{aligned} a_1 &:= a_{11} = a_{22} = \ldots = a_{nn} \\ a_2 &:= a_{12} = a_{23} = \ldots = a_{n-1,n} \\ &\vdots \\ a_k &:= a_{1k} = a_{2,k+1} = \ldots = a_{n-k+1,n} \\ a_{i,j} &:= 0, \ i > j. \end{aligned} \qquad \begin{aligned} m_1 &:= m_{11} = m_{22} = \ldots = m_{nn} \\ m_2 &:= m_{12} = m_{23} = \ldots = m_{n-1,n} \\ &\vdots \\ m_k &:= m_{1k} = m_{2,k+1} = \ldots = m_{n-k+1,n} \\ m_{i,j} &:= 0, \ i > j. \end{aligned}$$

Now, $am = 0$ implies

$$\sum_{i+j=t} a_i m_j = 0 \ for \ t = 2, 3, \ldots, k+1. \tag{1}$$

Since $_RM$ is a reduced module, from equation (1) we have that $a_1 m_1 = 0$ implies $a_1 R m_1 = 0$. By multiplying a_1 from the left to $a_1 m_2 + a_2 m_1$ (for $t = 3$), we obtain $a_1^2 m_2 = 0$, which implies $a_1 R m_2 = 0$. Consequently, this leads to $a_2 R m_1 = 0$. Similarly, by continuing this process for $t = 4, 5, \ldots, k+1$, we arrive at

$$a_i R m_j = 0 \ \forall \ i + j \leq k+1. \tag{2}$$

Again from $am = 0$, we have

$$\begin{aligned} a_1 m_{1,k+1} + a_2 m_k + a_3 m_{k-1} + \cdots + a_k m_2 + a_{1,k+1} m_1 &= 0, \\ a_1 m_{2,k+2} + a_2 m_k + a_3 m_{k-1} + \cdots + a_k m_2 + a_{2,k+2} m_1 &= 0, \\ &\vdots \\ a_1 m_{k+1,2k+1} + a_2 m_k + \cdots + a_{k-1} m_3 + a_k m_2 + a_{k+1,2k+1} m_1 &= 0. \end{aligned}$$

By applying the same process of left multiplications and using the earlier results obtained in equation 2, we conclude that for $u = 1, 2, \ldots, k+1$,

$$a_1 R m_{u,k+u} = a_{u,k+u} R m_1 = 0 \tag{3}$$

and with $i + j = k+2$ for i, j,

$$a_i R m_j = 0. \tag{4}$$

Now, for some $1 \leq l \leq k$, assume the condition $[a.m]_{u,k+u+t} = 0$ holds true for $t = 0, 1, \ldots, l-1$ and $u = 1, \ldots, k-t+1$. Thus, it is sufficient to show that for each $u = 1, \ldots, k-t+1$, the equation $[a.m]_{u,k+u+l} = 0$ holds true. For these, consider $a.m = 0$. This implies

$$\textstyle\sum_{j=1}^{n} a_{u,j} m_{j,k+u+l} = 0 \text{ for } u = 1, \ldots, k-l+1.$$

Thus,

$$\begin{aligned} a_1 m_{u,k+u+l} + \cdots + a_{l+1} m_{u+l,k+u+l} + a_{l+2} m_k + \cdots + a_k m_{l+2} + a_{u,k+u} m_{l+1} \\ + \cdots + a_{u,k+u+l-1} m_2 + a_{u,k+u+l} m_1 = 0. \end{aligned} \tag{5}$$

Again, by induction hypothesis and using results obtained in equations 2-4, we obtain the following:

(i) (a) $a_1Rm_{u,k+u+t} = a_{u,k+u+t}Rm_1 = 0$, for $u = 1, 2, \dots, k-t+1; t = 0, 1, \dots, l-1$.

(b) $a_2Rm_{u+1,k+u+t} = a_{u,k+u+t-1}Rm_2 = 0$, for $u = 1, 2, \dots, k-t+1; t = 1, \dots, l-1$.

$\vdots$

(c) $a_{t+1}Rm_{u+t,k+u+t} = a_{u,k+u}Rm_{l+1} = 0$, for $u = 1, 2, \dots, k-t+1$; $t = l-1$.

(ii) $a_iRm_j = 0$, for $i + j = u + k$, $i, j \geq u$ and $1 \leq u \leq l+1$.

Thus, statements (i) and (ii), along with the left multiplication process, imply that each left-side component of equation (5) equal to zero. Consequently, we have $[a.m]_{u,k+u+t} = 0$ for $u = 1, \dots, k-l+1$. By applying mathematical induction, it follows that $[a.m] = 0$ for all pairs (i, j). Therefore, for any $r \in A_n(R)$, we find that $arm = 0$. This implies that ${}_{A_n(R)}A_n(M)$ is semicommutative. □

Corollary 2.9. *Let R be a Reduced ring. For $n = 2k+1 \geq 3$, the $A_n(R)$ is a semicommutative.*

Corollary 2.10. *(Proposition 1.2, Kim) Let R be a reduced ring, then*

$$S = \left\{ \begin{pmatrix} r_1 & r_2 & r_3 \\ 0 & r_1 & r_4 \\ 0 & 0 & r_1 \end{pmatrix} : r_1, r_2, r_3, r_4 \in R \right\}$$

is a semicommutative.

Theorem 2.11. *Let ${}_RM$ be a Reduced module. For $n = 2k \geq 4$, the module $A_n(M) + E_{1,k}M$ is semicommutative over $A_n(R) + E_{1,k}R$.*

Proof. The proof of this theorem is almost similar to that of Theorem 2.8. However, for further illustration, we demonstrate it as follows: Let $a = [a_{ij}] \in A_n(R)$ and $m = [m_{ij}] \in A_n(M)$ satisfy $am = 0$. Firstly, notice that $a = [a_{ij}]$ and $m = [m_{ij}]$ have following properties:

$$\begin{aligned} a_1 &:= a_{11} = a_{22} = \dots = a_{nn} \\ a_2 &:= a_{12} = a_{23} = \dots = a_{n-1,n} \\ &\vdots \\ a_k &:= a_{1k} = a_{2,k+1} = \dots = a_{k+1,n} \\ a_0 &= a_{1,k} \\ a_{i,j} &:= 0, \ i > j \end{aligned} \qquad \begin{aligned} m_1 &:= m_{11} = m_{22} = \dots = m_{nn} \\ m_2 &:= m_{12} = m_{23} = \dots = m_{n-1,n} \\ &\vdots \\ m_k &:= m_{1k} = m_{2,k+1} = \dots = m_{k+1,n} \\ m_0 &= m_{1,k} \\ m_{i,j} &:= 0, \ i > j \end{aligned}$$

Now, we have

$$\sum_{i+j=t} a_im_j = 0 \ for \ t = 2, 3, \dots, k \tag{6}$$

and

$$a_1m_0 + a_2m_{k-1} + \dots + a_{l-1}m_2 + a_0m_1 = 0 \ \forall \ l = 1, 2, \dots, k. \tag{7}$$

By applying similar left multiplication to Equations 6 and 7, we obtain

$$a_iRm_j = 0 \ \forall \ i + j \leq k+1 \tag{8}$$

and

$$a_0Rm_1 = a_1Rm_0 = 0. \tag{9}$$

Again, from $am = 0$, we have

$$\begin{aligned} a_1m_{1,k+1} + a_2m_k + a_3m_{k-1} + \cdots + a_km_2 + a_{1,k+1}m_1 &= 0,\\ a_1m_{2,k+2} + a_2m_k + a_3m_{k-1} + \cdots + a_km_2 + a_{2,k+2}m_1 &= 0,\\ &\vdots\\ a_1m_{k,2k} + a_2m_k + \cdots + a_{k-1}m_3 + a_km_2 + a_{k,2k}m_1 &= 0. \end{aligned}$$

By applying the same process of left multiplications and using the earlier results obtained in Equations 6-9, we conclude that for $u = 1, 2, \dots, k + 1$,

$$a_1Rm_{u,k+u} = a_{u,k+u}Rm_1 = 0, \tag{10}$$

and with $i + j = k + 2$ for i, j

$$a_iRm_j = 0, \tag{11}$$

and

$$a_0Rm_2 = a_2Rm_0 = 0. \tag{12}$$

Now, for some $1 \leq l \leq k$, assume that the condition $[a.m]_{u,k+u+t} = 0$ holds true for $t = 0, 1, \dots, l$ and $u = 1, \dots, k-t$. Thus, it is sufficient to show that for each $u = 1, \dots, k-l$, the equation $[a.m]_{u,k+u+l} = 0$ holds true. For these, consider $a.m = 0$. This implies

$$\begin{cases} a_1m_{u,k+u+l} + \cdots + a_{l+1}m_{u+l,k+u+l} + a_{l+2}m_k + \cdots + a_km_{l+2} + a_{u,k+u}m_{l+1} + \\ \ldots + a_{u,k+u+l-1}m_2 + a_{u,k+u}m_{l+1} + \cdots + a_{u,k+u+l-1}m_2 + a_{u,k+u+l}m_1 = 0. \end{cases} \tag{13}$$

and

$$\begin{cases} a_1m_{1,k+l+1} + \cdots + a_{l+1}a_{l+1,k+1} + m_{l+2}g_k + \cdots + a_{k-1}m_{l+3} + a_0m_{l+2} + a_{1,k+1}m_{l+1} + \\ \ldots + a_{1,k+1}m_2 + a_{1,k+l+1}m_1 = 0. \end{cases} \tag{14}$$

Again, by induction hypothesis and using results obtained in Equations 7-12, we obtain the following:

(i) (a) $a_1Rm_{u,k+u+t} = a_{u,k+u+t}Rm_1 = 0$, for $u = 1, 2, \dots, k-t$; $t = 0, 1, \dots, l-1$.

(b) $a_2Rm_{u+1,k+u+t} = a_{u,k+u+t-1}Rm_2 = 0$, for $u = 1, 2, \dots, k-t+1$; $t = 1, \dots, l-1$.

$\vdots$

(c) $a_{t+1}Rm_{u+t,k+u+t} = a_{u,k+u}Rm_{t+1} = 0$, for $u = 1, 2, \dots, k-t$;$t = l-1$.

(ii) $a_iRm_j = 0$ for $i + j = u + k$, $i, j \geq u$ for $u = 1, 2, \dots, l + 1$.

(iii) $a_0m_u = 0$ for $u = 1, 2, \dots, l + 1$.

Thus, from (i), (ii), (iii) and the left multiplication process, we find that each component of Equations 13 and 14 equal to zero. Hence, $[a.m]_{u,k+u+t} = 0$ for $u = 1, \dots, k-l$. Hence, mathematical induction gives $[a.m] = 0$ for all (i, j). Therefore, for any $r \in A_n(R)$, we find that $arm = 0$. This implies that ${}_{A_n(R)}A_n(M)$ semicommutative. □

Let ${}_RM$ be a reduced module. Consider the following module

$$M_n = \left\{ \begin{pmatrix} v & v_{12} & v_{13} & \ldots & m_{1n} \\ 0 & v & v_{23} & \ldots & v_{2n} \\ 0 & 0 & v & \ldots & v_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & v \end{pmatrix} : v, v_{ij} \in M \right\}.$$

Based on Theorems 2.8 and 2.11, one might suspect that M_n is a semicommutative module over R_n for $n \geq 3$. However, the example provided below will dispel this possibility.

Example 2.12. *Consider a module ${}_RR$. Then we define*

$$R_n = \left\{ \begin{pmatrix} u & u_{12} & u_{13} & u_{14} \\ 0 & u & u_{23} & u_{24} \\ 0 & 0 & u & u_{34} \\ 0 & 0 & 0 & u \end{pmatrix} : u, u_{ij} \in R \right\}.$$

Now, we observe that.

$$\begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0,$$

but

$$\begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0.$$

Proposition 2.13. *Let ${}_RM$ be a reduced module. The following statements hold:*

(1) For $n = 2k+1 \geq 3$, the module $A_n(M)$ is strongly semicommutative over $A_n(R)$.

(2) For $n = 2k \geq 4$, the module $A_n(M) + E_{1,k}M$ is strongly semicommutative over $A_n(R) + E_{1,k}R$.

Proof. The proof is analogous to that of Theorem 2.8 and Theorem 2.11. □

Corollary 2.14. *Consider the following statements for a module ${}_RM$:*

(1) ${}_RM$ is reduced.

(2) $A_n(M)$ is a strongly semicommutative over $A_n(R)$ for $n = 2k+1 \geq 3$.

(3) $A_n(M) + E_{1,k}(M)$ is a strongly semicommutative over $A_n(R) + E_{1,k}(R)$ for $n = 2k \geq 4$.

(4) $A_n(M)$ is a semicommutative over $A_n(R)$ for $n = 2k+1 \geq 3$.

(5) $A_n(M) + E_{1,k}(M)$ is a semicommutative over $A_n(R) + E_{1,k}(R)$ for $n = 2k \geq 4$.

(6) $V_n(M)$ is a strongly semicommutative over $V_n(M)$ for $n \geq 2$.

(7) $\dfrac{M[x]}{(x^n)M[x]}$ is a strongly semicommutative over $\dfrac{R[x]}{(x^n)}$ for $n \geq 2$.

(8) $V_n(M)$ is a semicommutative over $V_n(R)$ for $n \geq 2$.

(9) $\dfrac{M[x]}{(x^n)M[x]}$ is semicommutative over $\dfrac{R[x]}{(x^n)}$ for $n \geq 2$.

Then the following implications are true:

(a) (1) $\Rightarrow$ (2) $\Rightarrow$ (4).

(b) (1) ⇒ (3) ⇒ (5).

(c) (1) ⇒ (2) + (3) ⇒ (6) ⇔ (7) ⇒ (8) ⇔ (9).

Proof.

(a) Follows from Theorem 2.8.

(b) Follows from Theorem 2.11 and Proposition 2.13.

(c) Follows from (a) , (b) of Corollary 2.14 and [16]. □

Recall that an element v in the module ${}_RM$ is considered torsion if there exists a non-zero element u in R such that $uv = 0$. We denote the set of all torsion elements in ${}_RM$ as $Tor(M)$. If R is a commutative domain, then $Tor(M)$ forms a submodule, and the quotient $M/Tor(M)$ is torsion-free. However, this is not the case if R contains a non-zero zero divisor. This can be illustrated using the example of $M = R = \mathbb{Z}_2 \times \mathbb{Z}_2$. In this case, we have $Tor(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{(0,0),(1,0),(0,1)\}$, which is not a submodule. Lastly, we have identified certain conditions for the strongly semicommutative property in the context of the torsion class.

Proposition 2.15. *Let ${}_RM$ be a weakly semicommutative module over a domain R. If $Tor(M)$ is strongly semicommutative, then ${}_RM$ is also a strongly semicommutative module.*

Proof. Since ${}_RM$ is weakly semicommutative, according to Proposition 3.6 in [1], we can conclude that $Tor(M)$ is a submodule of ${}_RM$. Now, let $m(x) = \sum_{k=0}^{q} v_k x^k \in M[x]$ and $p(x) = \sum_{l=0}^{n} u_l x^l \in R[x]$ such that $p(x)m(x) = 0$. We can expand this product, resulting in the following system of equations:

$$\begin{aligned} u_0v_0 &= 0, \\ u_1v_0 + u_0v_1 &= 0, \\ u_2v_0 + u_1v_1 + u_0v_2 &= 0, \\ &\vdots \\ u_nv_q &= 0. \end{aligned}$$

Assuming that $u_0 \neq 0$, we conclude from the first equation that $v_0 \in Tor(M)$. Next, by multiplying the second equation by u_0, we obtain $u_0^2 v_1 = 0$. Since R is a domain, this implies $v_1 \in Tor(M)$. Similarly, by multiplying the equation by u_0^2, we find $u_0^3 v_2 = 0$, which leads to $v_2 \in Tor(M)$. Continuing this process for a finite number of steps, we conclude that $m(x) \in Tor(M)[x]$. Thus, the assumption that $Tor(M)$ is a strongly semicommutative module implies that ${}_RM$ is also a strongly semicommutative module.

□

Corollary 2.16. *Let ${}_RM$ be a semicommutative module over a domain R. If $Tor(M)$ is strongly semicommutative, then ${}_RM$ is also a strongly semicommutative module.*

Corollary 2.17. *Let R be a commutative domain. The following conditions for a module ${}_RM$ are equivalent:*

(1) ${}_RM$ is strongly semicommutative.

(2) $Tor(M)$ is strongly semicommutative.

Proposition 2.18. *Let R be a domain. If ${}_RM$ is strongly semicommutative module, then the quotient module $M/Tor(M)$ is strongly semicommutative over R.*

Proof. Let $p(x) = \sum_{l=0}^{n} u_l x^l \in R[x]$ and $\overline{m(x)} = \sum_{k=0}^{q} \overline{v}_k x^k \in M/Tor(M)[x]$ satisfy $p(x)\overline{m(x)} = \overline{0}$ in $M/Tor(M)[x]$. Then, we have $\sum_{l+k=t} u_l \overline{v}_k = \overline{0}$ for $t = 0, 1, \ldots, n+k$. Thus, $\sum_{i+j=t} a_i m_j \in Tor(M)$ for $t = 0, 1, \ldots, n+q$. Hence, there exists $r_t \in R$ for $0 \leq t \leq n+k$ such that $r_t \sum_{l+k=t} u_l v_k = 0$. Let us take $h = \prod_{t=0}^{n+k} r_t$, and since ${}_RM$ is strongly semicommutative, we get $h \sum_{l+k=t} u_l v_k = 0$ for $0 \leq t \leq n+q$. Define $g(x) = hr(x)$. Clearly, $g(x) \in R[x]$, and since ${}_RM$ is a semicommutative module, we have $g(x)m(x) = 0$. This implies $g(x)R[x]m(x) = 0$. Since R is a domain, it implies $r(x)R[x]\overline{m(x)} = 0$. Hence, $M/Tor(M)$ is a strongly semicommutative module. ☐

Proposition 2.19. *Let $N \subseteq {}_RM$ be a reduced submodule. If $\dfrac{M}{N}$ is strongly semicommutative over R, then ${}_RM$ is strongly semicommutative.*

Proof. Since N is a reduced submodule, it follows that $N[x]$ is reduced as well, according to [14]. Furthermore, since $\dfrac{M[x]}{N[x]} \cong \dfrac{M}{N}[x]$ is semicommutative, we conclude that ${}_RM$ is strongly semicommutative. ☐

Recall that for a multiplicative closed subset S of the center C of the ring R, the set $S^{-1}M$ has a left module structure over $S^{-1}R$. In the next proposition, we study localization.

Proposition 2.20. *For a module ${}_RM$, the following conditions are equivalent.*

(1) ${}_RM$ is strongly semicommutative.

(2) $S^{-1}M$ is strongly semicommutative $S^{-1}R$-module for each multiplicatively closed subset S of C.

Proof. (1)⇒(2)Suppose the condition $p(x).m(x) = 0$ holds for some $p(x) = \sum_{i=0}^{m} \xi_i x^i \in S^{-1}R[x]$ and $m(x) = \sum_{j=0}^{n} \eta_j x^j \in S^{-1}M[x]$. Here $\xi_i = s_i^{-1} x^i \in S^{-1}R$ and $\eta_j = t_j^{-1} m_j \in S^{-1}M$. Thus we have

$$\begin{cases} \xi_0\eta_0 & = 0 \\ \xi_0\eta_1 + \xi_1\eta_0 & = 0, \\ \vdots & \\ \xi_m\eta_n & = 0. \end{cases}$$

Choose $s = (s_0 s_1 \ldots s_m)$ and $t = (t_0 t_1 \ldots t_n)$ in S and consider $\widehat{p(x)} = s.p(x) \in R[x]$, $\widehat{m(x)} = tm(x) \in M[x]$. Clearly $\widehat{p(x)}.\widehat{m(x)} = 0$. By choosing $h(x) = \sum_{k=0}^{l} \gamma_k x^k$ in $S^{-1}R[x]$ where $\gamma_k = u_k^{-1} x^k \in S^{-1}R$. then $\widehat{h(x)} = u.h(x) \in R[x]$, where $u = (u_0 u_1 \ldots u_k)$. Since ${}_RM$ is strongly semicommutative module implies $\widehat{p(x)}.\widehat{h(x)}.\widehat{m(x)} = 0$. Hence we see that $p(x)h(x)m(x) = 0$ since all s_i, t_j, u_k are central. Thus $S^{-1}M$ is strongly semicommutative module over $S^{-1}R$. (2)⇒(1) is obvious. ☐

Lee and Zhou, in [11], studied the polynomial extension of the Armendariz and reduced properties for a module ${}_RM$. Recall from [11], that ${}_RM$ being Armendariz (reduced) implies ${}_{R[x]}M[x]$ is also Armendariz (reduced) and vice-versa. However, Example 2.3 in this article demonstrates that the same is not true for the semicommutative property. Thus, it is of interest to consider the polynomial extension of the SSC property. The next proposition will validate this consideration. For a module ${}_RM$, recall that under usual addition and scalar multiplication, the Laurent polynomial extension

$$M[x, x^{-1}] := \{\textstyle\sum_{i=-s}^{t} m_i x^i : \ s \geq 0, \ t \geq 0, \ m_i \in M\} \text{ is a module over } R[x, x^{-1}].$$

Proposition 2.21. *The following conditions are equivalent for a module ${}_RM$:*

(1) The module ${}_RM$ is strongly semicommutative.

(2) The module ${}_{R[x]}M[x]$ is strongly semicommutative.

(3) The module ${}_{R[x,\,x^{-1}]}M[x,\,x^{-1}]$ is strongly semicommutative.

Proof. (1) ⇒ (2): consider $f(y) = \sum_{i=0}^{n} a_i(x)y^i \in R[x][y]$ and $m(y) = \sum_{j=0}^{t} m_j(x)y^j \in M[x][y]$ are such that $f(y)m(y) = 0$ and $h(y) = \sum_{w=0}^{e} b_w(x)y^w \in R[x][y]$. Here $a_i(x) = a_{i0} + a_{i1}x + \cdots + a_{ik_i}x^{k_i}$ and $m_j(x) = m_{j0} + m_{j1}x + \cdots + m_{jl_j}x^{l_j}$ and $b_w(x) = b_{w0} + b_{w1}x + \cdots + b_{wr_w}x^{r_w}$. Then from $f(y)m(y) = 0$, we have

$$\begin{aligned} a_0(x)m_0(x) =&0,\\ a_0(x)m_1(x) + a_1(x)m_0(x) =&0,\\ a_0(x)m_2(x) + a_1(x)m_1(x) + a_2(x)m_0(x) =&0,\\ &\vdots\\ a_n(x)m_t(x) =&0. \end{aligned}$$

Let $q \in \mathbb{N}$ be such that $q > d(a_0(x)) + d(a_1(x)) + \cdots + d(a_n(x)) + d(b_0(x)) + d(b_1(x)) + \cdots + d(b_e(x)) + d(m_0(x)) + d(m_1(x)) + \cdots + d(m_t(x))$, where $d(a_i(x))$ and $d(m_j(x))$ represent the degree of the polynomial. Then consider $\widehat{f(x)} = a_0(x^s) + a_1(x^s)x^{sq+1} + a_2(x^s)x^{2sq+2} + \cdots + a_n(x^s)x^{nsq+n}$, $\widehat{h(x)} = b_0(x^s) + b_1(x^s)x^{sq+1} + b_2(x^s)x^{2sq+2} + \cdots + b_e(x^s)x^{esq+e}$ and $\widehat{m(x)} = m_0(x^s) + m_1(x^s)x^{sq+1} + m_2(x^s)x^{2sq+2} + \cdots + m_t(x^s)x^{tsq+t}$. Then we have

$$\begin{aligned} \widehat{f(x)} =& a_{00} + a_{01}x^s + a_{02}x^{2s} + \cdots + a_{0k_0}x^{sk_0}\\ &+ a_{10}x^{qs+1} + a_{11}x^{qs+q+1} + a_{12}x^{qs+2q+1} + \cdots + a_{0k_1}x^{sq+qk_1+1}\\ &+\cdots\\ &+ a_{n0}x^{qs+n} + a_{n1}x^{nqs+q+n} + a_{n2}x^{nqs+2q+n} + \cdots + a_{0k_n}x^{nsq+qk_n+n} \end{aligned}$$

$$\begin{aligned} \widehat{h(x)} =& b_{00} + b_{01}x^s + b_{02}x^{2s} + \cdots + b_{0r_0}x^{sr_0}\\ &+ b_{10}x^{qs+1} + b_{11}x^{qs+q+1} + b_{12}x^{qs+2q+1} + \cdots + b_{0r_1}x^{sq+qr_1+1}\\ &+\cdots\\ &+ a_{e0}x^{qs+e} + b_{e1}x^{eqs+q+e} + b_{e2}x^{eqs+2q+e} + \cdots + b_{0r_e}x^{esq+qr_e+e} \end{aligned}$$

and

$$\begin{aligned} \widehat{m(x)} =& m_{00} + m_{01}x^s + m_{02}x^{2s} + \cdots + a_{0l_0}x^{sl_0}\\ &+ m_{10}x^{qs+1} + m_{11}x^{qs+q+1} + m_{12}x^{qs+2q+1} + \cdots + m_{0l_1}x^{sq+ql_1+1}\\ &+\cdots\\ &+ m_{t0}x^{qs+t} + m_{t1}x^{tqs+q+t} + m_{t2}x^{tqs+2q+t} + \cdots + m_{0l_t}x^{tsq+ql_t+t}. \end{aligned}$$

Using the above equations, we can see that $\widehat{f(x)}\widehat{m(x)} = 0$ in ${}_{R[x]}M[x]$. Since M is a strongly semicommutative module, it implies $\widehat{f(x)}\widehat{h(x)}\widehat{m(x)} = 0$. Thus we can conclude that $f(y)h(y)m(y) = \widehat{f(x)}\widehat{h(x)}\widehat{m(x)} = 0$. Hence, ${}_{R[x]}M[x]$ is a strongly semicommutative module. (2)⇒(3) The proof of this follows from Proposition 2.20 given above. (3)⇒(1) The proof is straightforward, as being a submodule, ${}_RM$ is a strongly semicommutative module. □

3. Concluding remarks

In this section, we present some conclusions and questions that arose from the study:

(1) Consider the following conditions for a module ${}_RM$:

(a) ${}_RM$ is a reduced module.

(b) ${}_RM$ is an Armendariz module.

(c) ${}_RM$ is a strongly semicommutative module.

(d) ${}_RM$ is a semicommutative module.

(e) ${}_RM$ is abelian.

For the general setting, we have the following implications:

(i) (a) $\Rightarrow$ (b) $\Rightarrow$ (e),

(ii) (a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e),

(iii) (a) $\Rightarrow$ (b) + (d) $\Rightarrow$ (c) $\Rightarrow$ (e).

The converses of (i), (ii), and (iii) are not valid in general, with sufficient counterexamples available in the literature. However, if ${}_RM$ is *Principally Projective*(a module ${}_RM$ called *Principally Projective* if the right annihilator of every element $m \in M$ in R is generated by an idempotent i.e., $r_R(m) = eR$ for all $m \in {}_RM$), then (e) $\Rightarrow$ (a), (b), (c), (d) (by [11]). By Theorem 2.11, Proposition 2.13, and Example 2.8, we can conclude that there are ample examples of modules that are semicommutative but not strongly semicommutative, indicating that the class of strongly semicommutative modules is distinct.

(2) For a polynomial module ${}_{R[x]}M[x]$, consider the following extensions:

$$M[[x]] := \{\textstyle\sum_{k=0}^{\infty} v_k x^k : v_k \in M\},$$
$$M[[x,x^{-1}]] := \{\textstyle\sum_{k=-t}^{\infty} v_k x^k : t \geq 0, v_k \in M\},$$

These are known as the power series and Laurent series extensions of ${}_RM$, respectively. It is routine to verify that ${}_{R[[x]]}M[[x]]$ and ${}_{R[[x,x^{-1}]]}M[[x,x^{-1}]]$ are modules under the usual addition and scalar multiplication. From our study, the following implications are evident: ${}_RM$ is strongly semicommutative $\Leftrightarrow$ ${}_{R[x]}M[x]$ is strongly semicommutative $\Leftrightarrow$ ${}_{R[x,x^{-1}]}M[x,x^{-1}]$. Given this background, it is natural to consider the following questions:

(a) If ${}_RM$ is a strongly semicommutative, does this imply that ${}_{R[[x]]}M[[x]]$ is also a strongly semicommutative?

(b) If ${}_{R[x,x^{-1}]}M[x,x^{-1}]$ is a strongly semicommutative, does this imply that ${}_{R[[x,x^{-1}]]}M[[x,x^{-1}]]$ is also strongly semicommutative?

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