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A note on CII groups and CCII groups*

Research Article

Teerapong Suksumran

Abstract: A group G is CII or, equivalently, 2-Engel if $[g, h] = [g^{-1}, h^{-1}]$ for all elements g and h in G, and is CCII if the central quotient $G/Z(G)$ is CII. In this paper, we give sufficient conditions and necessary conditions for a group to be CCII. In particular, we show that every CCII group is nilpotent of class at most 4 and list all CII groups and all CCII groups of order n with $n < 64$ up to isomorphism.

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Keywords: Nilpotent group, CCII group, CII group, 2-Engel group, Gyrogroup

1. Introduction

The notion of nilpotent groups is of great importance in the classification of groups, especially the classification of Lie groups. Furthermore, nilpotent groups and n-Engel groups are closely related in various ways. In fact, several well-known groups such as finite p-groups and the Heisenberg group turn out to be nilpotent. In [12, 15], the authors introduce two families of groups, which have strong connections with 2-Engel groups and nilpotent groups, in order to increase explicit examples of a non-associative structure called a gyrogroup. They also list all finite groups in these families of order less than 32 using the classification of groups of order less than 32 in Appendix B of [8]. In the present paper, we continue this work. In fact, the goal of this paper is twofold: to investigate the family of CII groups and the family of CCII groups (see Section 3 for the relevant definitions) and to list all CII groups and all CCII groups of order less than 64 up to isomorphism. It turns out that CII groups and CCII groups are all nilpotent, and some of which give rise to non-degenerate gyrogroups (see Section 6). This fact emphasizes the importance of nilpotent groups in the theory of gyrogroups.

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1Teerapong Suksumran; Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand (email: teerapong.suksumran@cmu.ac.th).

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2. Preliminaries

We follow standard terminology and notation in group theory. The basic theory of nilpotent groups can be found in, for instance, [5]. Here, we mention notations and useful results used in the sequel. Let G be a group. The center of G is denoted by $Z(G)$. For each pair of elements $g, h \in G$, the commutator of (g, h) is denoted by $[g, h]$ and is defined as $[g, h] = g^{-1}h^{-1}gh$. In the case when G is nilpotent, the nilpotency class of G is denoted by $\mathfrak{n}(G)$. We state the following well-known fact for easy reference.

Proposition 2.1. *Let p be a prime, and let k be an integer such that $k \geq 2$. If G is a group of order p^k, then G is nilpotent and $\mathfrak{n}(G) \leq k - 1$.*

3. Properties of CII groups and CCII groups

Recall that a group G is said to be *commutator-inversion invariant* (abbreviated to *CII*) or, equivalently, *2-Engel* if $[g, h] = [g^{-1}, h^{-1}]$ for all $g, h \in G$ (see Theorem 3.1 of [12]). Also, recall that a group G is said to be *CCII* if its central quotient $G/Z(G)$ is CII (see Definition 1 of [15]). In this section, we collect basic properties of CII groups and CCII groups; some of which are deduced from Levi's results straightforwardly. Therefore, not all results are claimed to be new. For the sake of consistence, we use the term "CII" instead of "2-Engel". We first show that the family of CII groups is included in the family of CCII groups.

Proposition 3.1. *Let G be a group. If G is CII, then G is CCII.*

Proof. Suppose that G is CII. By Proposition 3.1 of [12], $G/Z(G)$ is CII. By definition, G is CCII. □

The converse of Proposition 3.1 is not, in general, true. In fact, the dihedral group D_{16} is not CII, but its central quotient is CII (see page 6 of [12]). Hence, D_{16} is CCII. This in particular shows that the family of CII groups is properly included in the family of CCII groups. Moreover, we will see later that the dihedral group D_{2n} is not CCII for all odd integers $n \geq 3$. Next, we exhibit several sufficient conditions and necessary conditions for a group to be CII or CCII. Using Levi's result, we obtain the following theorem immediately.

Theorem 3.2. *Every group of exponent 3 is CII.*

Proof. By Levi's result [9], every group of exponent 3 is 2-Engel. Hence, the theorem follows immediately from the fact that a group is CII if and only if it is 2-Engel. □

In light of Theorem 3.2, the Burnside group $B(m, 3)$ is CII for all integers $m \geq 3$. Furthermore, as mentioned in Corollary 3.1 of [12], every nilpotent group of class at most 2 is CII. We summarize relationships between some classes of groups in Figure 1. The next theorem is a partial converse of Corollary 3.1 of [12] in the case of finite groups.

Theorem 3.3. *Every finite CII group without elements of order 3 is nilpotent of class at most 2.*

Proof. According to Burnside's result [3], every finite 2-Engel group without elements of order 3 is nilpotent of class at most 2. Hence, the theorem follows immediately. □

In light of Theorem 3.3, the property of being CII and the property of being nilpotent of class at most 2 are equivalent in the case of finite groups without elements of order 3. Therefore, we obtain the following corollary as an application of Cauchy's Theorem.

Corollary 3.4. *Suppose that G is a finite group whose order is not divisible by 3. Then G is CII if and only if G is nilpotent of class at most 2.*

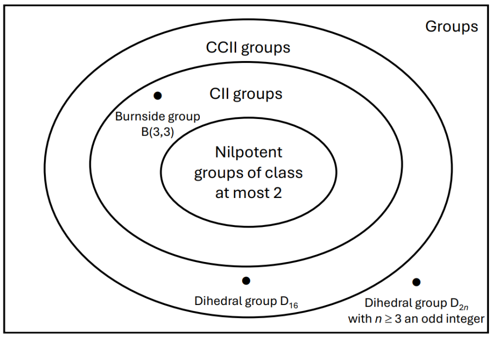

Figure 1. Relationships between nilpotent groups of class at most 2, CII groups, and CCII groups.

It turns out that the family of CII groups is included in the family of nilpotent groups of class at most 3, as stated in the following theorem.

Theorem 3.5. *Every CII group is nilpotent of class at most* 3*.*

Proof. According to Levi's result [9], every 2-Engel group is nilpotent of class at most 3. Hence, the theorem follows immediately. □

We remark that there exists a nilpotent group of class 3 that is not CII. In fact, the dihedral group D_{16} is nilpotent of class 3 but is not CII. This in particular shows that the family of CII groups is properly included in the family of nilpotent groups of class at most 3. Also, since the center of a nilpotent group cannot be trivial, it follows that a non-trivial group with trivial center cannot be CII as a consequence of Theorem 3.5. Recall that a group is said to be *centerless* if its center is trivial.

Proposition 3.6. *If G is a non-trivial centerless group, then G is neither CII nor CCII.*

Proof. Suppose that G is non-trivial and centerless. Then G is not nilpotent by Lemma 2.2 of [5]. Hence, G is not CII by Theorem 3.5. This also implies that G is not CCII because $G/Z(G) \cong G$. □

As an application of Proposition 3.6, the following groups are neither CII nor CCII:

- the symmetric group S_n for all $n \geq 3$;
- the dihedral group D_{2n} for all odd integers $n \geq 3$;
- the non-abelian simple groups;
- the Frobenius groups

because they are centerless. Using the fact that a non-trivial group G is nilpotent of class n if and only if $G/Z(G)$ is nilpotent of class $n-1$ (see Lemma 2.12 of [5]), we directly obtain the following theorem.

Theorem 3.7. *Every nilpotent group of class at most* 3 *is CCII.*

Proof. Suppose that G is nilpotent of class n with $n \leq 3$. In the case when $n = 0$, G is trivial and hence is CCII. Now, assume that $n \geq 1$. By Lemma 2.12 of [5], $G/Z(G)$ is nilpotent of class at most $n-1$. Thus, $G/Z(G)$ is nilpotent of class at most 2 and hence is CII. This shows that G is CCII. □

In light of Theorem 3.5 as well as Lemma 2.12 of [5], we immediately obtain the following theorem.

Theorem 3.8. *Every CCII group is nilpotent of class at most* 4.

Proof. Suppose that G is CCII. By definition, $G/Z(G)$ is CII. By Theorem 3.5, $G/Z(G)$ is nilpotent of class at most 3. Hence, by Lemma 2.12 of [5], G is nilpotent of class at most 4. □

We remark that there exists a nilpotent group of class 4 that is not CCII. In fact, the dihedral group D_{32} is nilpotent of class 4. However, D_{32} is not CCII because $D_{32}/Z(D_{32}) \cong D_{16}$ and D_{16} is not CII. This in particular shows that the family of CCII groups is properly included in the family of nilpotent groups of class at most 4. Moreover, Theorem 3.8 leads to a natural question whether a CCII group that is nilpotent of class 4 exists. This question has the affirmative answer. Recall that a group H is said to be *capable* if there exists a group G such that $H \cong G/Z(G)$. As in [2], a group is capable if and only if its epicenter is trivial. It can be checked by using GAP that there exists a capable CII group of nilpotency class 3 (see the appendix), and so a CCII group of nilpotency class 4 exists. We summarize some relationships between CII groups, CCII groups, and nilpotent groups in Figure 2. We close this section with a diagram to check whether a group is CII or CCII in Figure 3.

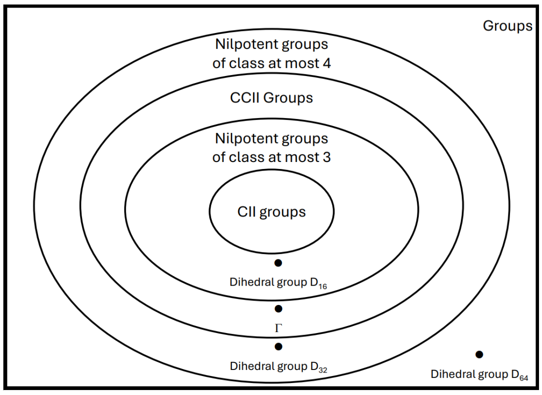

Figure 2. Relationships between CII groups, CCII groups, and nilpotent groups. Here, Γ is a group such that $\Gamma/Z(\Gamma)$ is a CII group of class 3.

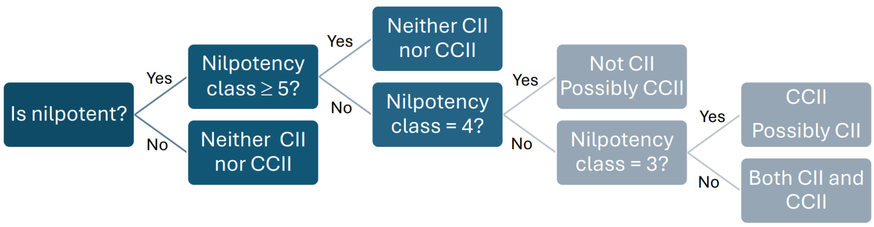

Figure 3. Diagram of verifying whether a group is CII or CCII.

4. Finite CII groups and finite CCII groups

In this section, we focus on finite CII groups and finite CCII groups. According to Theorems 3.5 and 3.8, every CII group and every CCII group are nilpotent. Therefore, we obtain the following sufficient and necessary conditions for a finite group to be CII and to be CCII, respectively, using the well-known property of finite nilpotent groups.

Theorem 4.1. *Let G be a finite group. Then G is CII if and only if G is isomorphic to a finite direct product of CII finite p-groups of nilpotency class at most* 3.

Proof. Suppose that G is CII. By Theorem 3.5, G is nilpotent of class at most 3. Hence, $G \cong P_1 \times P_2 \times \cdots \times P_k$, where P_i is a unique Sylow p_i-subgroup of G corresponding to each prime divisor p_i of $|G|$. Hence, P_i is CII for all i by part 1 of Proposition 3.4 of [12]. Furthermore, P_i is nilpotent of class at most 3 for all i. The converse holds immediately by part 1 of Proposition 3.4 of [12]. □

Theorem 4.2. *Let G be a finite group. Then G is CCII if and only if G is isomorphic to a finite direct product of CCII finite p-groups of nilpotency class at most* 4.

Proof. The proof can be done in a similar fashion to the proof of Theorem 4.1, using Theorem 3.8 and part 2 of Proposition 3.4 of [12]. □

Theorem 4.1 states that, in order to study finite CII groups, one may examine finite p-groups of nilpotency class at most 3. Similarly, Theorem 4.2 states that, in order to study finite CCII groups, one may examine finite p-groups of nilpotency class at most 4. The next lemma shows that a finite group whose order is cubic-prime-free is nilpotent if and only if it is abelian. This implies that a finite non-abelian group whose order is cubic-prime-free is neither CII nor CCII, as shown in Proposition 4.4.

Lemma 4.3. *Suppose that n is a positive integer written in the canonical form as $n = p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k}$, where $p_1 < p_2 < \cdots < p_k$ are primes and t_i is an integer with $1 \le t_i \le 2$ for $i = 1, 2, \ldots, k$. If G is a group of order n, then G is nilpotent if and only if G is abelian.*

Proof. Suppose that $|G| = n$. It is clear that if G is abelian, then G is nilpotent. Assume that G is nilpotent. Then $G \cong P_1 \times P_2 \times \cdots \times P_k$, where P_i is a unique Sylow p_i-subgroup of G for $i = 1, 2, \ldots, k$. Since $|P_i| \in \{p_i, p_i^2\}$, it follows that P_i is abelian for all i, and so G is abelian. □

Proposition 4.4. *Suppose that n is a positive integer written in the canonical form as $n = p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k}$, where $p_1 < p_2 < \cdots < p_k$ are primes and t_i is an integer with $1 \le t_i \le 2$ for $i = 1, 2, \ldots, k$. Then every non-abelian group of order n is neither nilpotent, CII, nor CCII.*

Proof. Suppose that G is non-abelian of order n. By Lemma 4.3, G is not nilpotent, and so G is neither CII nor CCII. □

The following lemma shows that if a finite group whose order is quartic-prime-free is nilpotent, then its nilpotency class cannot exceed 2. This implies that a finite nilpotent group whose order is quartic-prime-free is both CII and CCII, as shown in Proposition 4.6.

Lemma 4.5. *Suppose that n is a positive integer written in the canonical form as $n = p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k}$, where $p_1 < p_2 < \cdots < p_k$ are primes and t_i is an integer with $1 \le t_i \le 3$ for $i = 1, 2, \ldots, k$. If G is a group of order n, then G is nilpotent implies the nilpotency class of G is at most* 2.

Proof. Suppose that $|G| = n$. Assume that G is nilpotent. Then $G \cong P_1 \times P_2 \times \cdots \times P_k$, where P_i is a unique Sylow p_i-subgroup of G for $i = 1, 2, \ldots, k$. Since $|P_i| \in \{p_i, p_i^2, p_i^3\}$, it follows that $\mathfrak{n}(P_i) \le 2$. This implies that $\mathfrak{n}(G) \le 2$. □

Proposition 4.6. *Suppose that n is a positive integer written in the canonical form as $n = p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k}$, where $p_1 < p_2 < \cdots < p_k$ are primes and t_i is an integer with $1 \le t_i \le 3$ for $i = 1, 2, \ldots, k$. Then every nilpotent group of order n is CII and CCII.*

Proof. Suppose that G is nilpotent of order n. By Lemma 4.5, $\mathfrak{n}(G) \le 2$, and so G is CII and CCII. □

Since every group of order p^3, where p is a prime, is nilpotent, Proposition 4.6 yields the following consequence:

Corollary 4.7. *Let p be a prime. Then every group of order p^3 is CII and CCII.*

Next, we show that every group of order p^4, where p is a prime, is always CCII (but not necessarily CII).

Proposition 4.8. *Let p be a prime. Then every group of order p^4 is CCII.*

Proof. Suppose that $|G| = p^4$. It is known that G must be nilpotent of class at most 3 (see Proposition 2.1). Hence, Theorem 3.7 applies. □

Let p be a prime. We remark that there exists a group of order p^4 that is not CII (for example, D_{16}), and there exists a CII group of order p^4 (for example, $\mathbb{Z}_2 \times D_8$). Also, a classification of groups of order p^4, where $p > 2$, can be found in [4]. As mentioned in [1], if $p > 2$, then there are only four non-abelian groups of nilpotency class 3 (up to isomorphism):

1. $G_{p^4,7} = \langle a, b \mid a^p = b^p = [a, b]^p = [a, [a, b]]^p = [b, [a, b]] = e, [a, [a, [a, b]]] = [b, [a, [a, b]]] = e\rangle$;
2. $G_{p^4,8} = \langle a, b \mid a^{p^2} = b^p = [a, b]^p = [b, [a, b]] = e, [a, [a, b]] = a^p\rangle$;
3. $G_{p^4,9} = \langle a, b \mid a^{p^2} = b^p = [a, b]^p = [a, [a, b]] = e, [b, [a, b]] = a^p\rangle$;
4. $G_{p^4,10} = \langle a, b \mid a^{p^2} = b^p = [a, b]^p = [a, [a, b]] = e, [b, [a, b]] = a^{2p}\rangle$.

Note that $G_{p^4,9}$ and $G_{p^4,10}$ are not CII because $[b, [a, b]] \neq e$ (that is, they are not 2-Engel). Moreover, $G_{p^4,8}$ is not CII since otherwise $[a, [b, a]] = e$ would imply $e = a^{-1}[b, a]^{-1}a[b, a]$, which would imply $[a, b]a = a[a, b]$, and so $[a, [a, b]] = e$, contrary to the fact that $[a, [a, b]] = a^p$. Similarly, $G_{p^4,7}$ is not CII since otherwise $[a, [b, a]] = e$ would imply $[a, [a, b]] = e$, contrary to the fact that $[a, [a, b]]$ has order p. We remark that the remaining six non-abelian groups (up to isomorphism) are of nilpotency class 2 so that they are CII. Moreover, the only nilpotent groups of order 16 of class 3 are D_{16}, Q_{16}, and SD_{16}. Let us summarize this fact as Proposition 4.9.

Proposition 4.9. *Let G be a group of order p^4, where p is a prime. Then G is CII if and only if the nilpotency class of G is at most 2.*

For a group of order p^5, where p is a prime, we need to know its center in order to determine whether it is CCII in some circumstances.

Proposition 4.10. *Suppose that G is a group of order p^5, where p is a prime. If $Z(G)$ is not of order p, then G is CCII. If $Z(G)$ is of order p and the nilpotency class of G is at most 3, then G is CCII.*

Proof. By Lagrange's Theorem, $|Z(G)| \in \{1, p, p^2, p^3, p^4, p^5\}$. Since G is a finite p-group, $|Z(G)| \neq 1$. In the case when $|Z(G)| \in \{p^3, p^4, p^5\}$, $|G/Z(G)| \in \{1, p, p^2\}$, and so $G/Z(G)$ is abelian. In the case when $|Z(G)| = p^2$, $G/Z(G)$ has order p^3, which implies that $G/Z(G)$ is CII by Corollary 4.7. This shows that G is CCII. Now, suppose that $|Z(G)| = p$ and $\mathfrak{n}(G) \leq 3$. Then $G/Z(G)$ has order p^4 and $\mathfrak{n}(G/Z(G)) \leq 2$. It follows that $G/Z(G)$ is CII, and so G is CCII. □

Using a classification of groups of order 32 (see, for instance, [7]), we can refine the result in Proposition 4.10 and complete this section with the following proposition.

Proposition 4.11. *Suppose that G is a group of order 32. If $Z(G)$ is not of order 2, then G is CCII. If $Z(G)$ is of order 2, then G is CCII if and only if the nilpotency class of G is at most 3.*

Proof. In light of Proposition 4.10, we need only show that if $|Z(G)| = 2$ and G is CCII, then $\mathfrak{n}(G) \leq 3$. Suppose that $|Z(G)| = 2$ and G is CCII. Hence, $G/Z(G)$ has order 16 and is CII. This implies that $\mathfrak{n}(G/Z(G)) \leq 2$ since there are only three groups of order 16 that are not CII up to isomorphism (which are D_{16}, Q_{16}, and SD_{16}) and all of which have nilpotency class 3. It follows that $\mathfrak{n}(G) \leq 3$, and the proof completes. □

5. Tables of CII groups and CCII groups of small order

In [12, 15], the authors collect finite CCII groups of order less than 32 with the aid of classification of finite non-abelian groups, up to isomorphism, as in Appendix B of [8]. In this section, we refine this result by determining whether they are CII and continue to list CII groups and CCII groups of order less than 64. We follow the list of non-abelian groups of order n with $32 \leq n \leq 63$ exhibited in [6]. In Tables 1–7, which are built by applying the results in Section 4, $\mathbb{Z}_n$ denotes the group of integers modulo n; $\Lambda_n[m]$ denotes an abelian group of order n without specifying its isomorphism type, and m is the number of isomorphism types of Λ_n; S_n denotes the symmetric group of degree n; D_{2n} denotes the dihedral group of order $2n$; Q_{4n} denotes the generalized quaternion group (also called the dicyclic group) of order $4n$; A_n denotes the alternating group of degree n; SD_{2^n} denotes the semidihedral group (also called the quasidihedral group) of order 2^n; M_n denotes the modular maximal-cyclic group of order n; $\Gamma_{n,m}$ denotes the mth small group of order n in the Small Groups Library of GAP; $\mathrm{Dih}(A)$ denotes the generalized dihedral group of an abelian group A; F_n denotes the Frobenius group of order n; $SL(m, n)$ denotes the special linear group of $m \times m$ matrices with entries from a field of n elements; $B(m, n)$ denotes the Burnside group induced by a free group F_m of rank m and the (normal) subgroup generated by all nth powers of elements of F_m; $\mathrm{Hol}(G)$ denotes the holomorph of a group G; and $GL(m, n)$ denotes the general linear group of $m \times m$ matrices with entries from a field of n elements.

6. Gyrogroups associated with CCII groups

A gyrogroup is a group-like structure whose binary operation is, in general, non-associative. This type of algebraic structure arises from the study of parametrization of the Lorentz transformation group in [13]. For more details of formation of gyrogroup theory, we refer the reader to [14] and the references therein. For basic knowledge of gyrogroups, we refer the reader to [10, 14].

Here, we discuss close relationships between CCII groups and gyrogroups. The importance of CCII groups lies in the fact that any CCII group induces a gyrogroup, which enables us to have more examples of concrete gyrogroups, as shown in Theorem 4.2 of [12]. In fact, if G is a CCII group, then the underlying set of G can be made into a gyrogroup, denoted by G^{gyr}, under the binary operation defined by the formula

$$a \oplus b = aaba^{-1} \qquad \text{for all } a, b \in G. \tag{1}$$

In the resulting gyrogroup G^{gyr}, the identity of G^{gyr} is the same as the identity of G, and the inverse of an element a in G^{gyr} is the same as the inverse of a in G. Furthermore, if a and b are elements in G, then the gyroautomorphism of G^{gyr} generated by a and b is the inner automorphism generated by the commutator $[a^{-1}, b]$. It is proved in Theorem 4.3 of [12] that G^{gyr} is associative if and only if G is nilpotent of class at most 2, which gives a characterization for G^{gyr} to be a group under the induced gyrogroup operation. According to Proposition 4.2 of [12], if G and H are isomorphic CCII groups, then G^{gyr} and H^{gyr} are isomorphic as gyrogroups. The converse is not, in general, true. However, if the orders of G and H are not divisible by 3, then the converse holds. In light of the results in Section 5, we conclude that D_{16}^{gyr}, Q_{16}^{gyr}, and SD_{16}^{gyr} are pairwise non-isomorphic non-degenerate gyrogroups of order 16; $\Gamma_{32,6}^{\mathrm{gyr}}$, $\Gamma_{32,7}^{\mathrm{gyr}}$, $\Gamma_{32,8}^{\mathrm{gyr}}$, $\Gamma_{32,9}^{\mathrm{gyr}}$, $\Gamma_{32,10}^{\mathrm{gyr}}$, $\Gamma_{32,11}^{\mathrm{gyr}}$, $\Gamma_{32,13}^{\mathrm{gyr}}$, $\Gamma_{32,14}^{\mathrm{gyr}}$, $\Gamma_{32,15}^{\mathrm{gyr}}$, $\Gamma_{32,42}^{\mathrm{gyr}}$, $\Gamma_{32,44}^{\mathrm{gyr}}$, $\mathrm{Hol}(\mathbb{Z}_8)^{\mathrm{gyr}}$, $(\mathbb{Z}_2 \times D_{16})^{\mathrm{gyr}}$, $(\mathbb{Z}_2 \times SD_{16})^{\mathrm{gyr}}$, and $(\mathbb{Z}_2 \times Q_{16})^{\mathrm{gyr}}$ are pairwise non-isomorphic non-degenerate gyrogroups of order 32; and $(\mathbb{Z}_3 \times D_{16})^{\mathrm{gyr}}$, $(\mathbb{Z}_3 \times SD_{16})^{\mathrm{gyr}}$, and $(\mathbb{Z}_3 \times Q_{16})^{\mathrm{gyr}}$ are pairwise non-isomorphic non-degenerate gyrogroups of order 48 (as a consequence of Proposition 1 of [15] and Theorem 3.3 of [11]). In the case when p is a prime such that $p > 3$, we have an infinite series of pairwise non-isomorphic non-degenerate gyrogroups of order p^4: $G_{p^4,7}^{\mathrm{gyr}}$, $G_{p^4,8}^{\mathrm{gyr}}$, $G_{p^4,9}^{\mathrm{gyr}}$, and $G_{p^4,10}^{\mathrm{gyr}}$.

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Disclosure statement

Data Availability Statement: The authors declare that [the/all other] data supporting the findings of this study are available within the article. Any clarification may be requested from the corresponding author, provided it is essential.
Competing interests: The authors declare that there is no conflict of interest regarding the publication of this manuscript.

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Appendix

In the appendix, we exhibit an example of a command in GAP for verifying that there exists a capable CII group of class 3.

Example

```
gap> LoadPackage( "nq" );
true
gap> F := FreeGroup( "a","b","c","x" );
<free group on the generators [ a, b, c, x ]>
gap> AssignGeneratorVariables( F );
#I Assigned the global variables [ a, b, c, x ]
gap> G := NilpotentQuotient( F/[x^3]:  idgens := [x] );
Pcp-group with orders [ 3, 3, 3, 3, 3, 3, 3 ]
gap> IsNilpotent ( G );
true
gap> NilpotencyClassOfGroup( G );
3
gap> IsCII := function( G ) return ForAll( ConjugacyClasses( G ),
c -> IsAbelian ( NormalClosure ( G, Subgroup ( G,
> [Representative( c )] ) ) ) ); end;
function( G ) ...  end
gap> IsCII ( G );
true
gap> Epicenter ( G );
Pcp-group with orders [ ]
```

Order	Group	CII group?	CCII group?	Is nilpotent of class at most 2?
1	$\{e\}$	Yes	Yes	Yes
2	$\mathbb{Z}_2$	Yes	Yes	Yes
3	$\mathbb{Z}_3$	Yes	Yes	Yes
4	$\Lambda_4[2]$	Yes	Yes	Yes
5	$\mathbb{Z}_5$	Yes	Yes	Yes
6	$\mathbb{Z}_6$	Yes	Yes	Yes
6	S_3	No	No	No
7	$\mathbb{Z}_7$	Yes	Yes	Yes
8	$\Lambda_8[3]$	Yes	Yes	Yes
8	D_8	Yes	Yes	Yes
8	Q_8	Yes	Yes	Yes
9	$\Lambda_9[2]$	Yes	Yes	Yes
10	$\mathbb{Z}_{10}$	Yes	Yes	Yes
10	D_{10}	No	No	No
11	$\mathbb{Z}_{11}$	Yes	Yes	Yes
12	$\Lambda_{12}[2]$	Yes	Yes	Yes
12	A_4	No	No	No
12	D_{12}	No	No	No
12	Q_{12}	No	No	No
13	$\mathbb{Z}_{13}$	Yes	Yes	Yes
14	$\mathbb{Z}_{14}$	Yes	Yes	Yes
14	D_{14}	No	No	No
15	$\mathbb{Z}_{15}$	Yes	Yes	Yes
16	$\Lambda_{16}[5]$	Yes	Yes	Yes
16	D_{16}	No	Yes	No
16	Q_{16}	No	Yes	No
16	SD_{16}	No	Yes	No
16	$\mathbb{Z}_2 \times D_8$	Yes	Yes	Yes
16	$\mathbb{Z}_2 \times Q_8$	Yes	Yes	Yes
16	M_{16}	Yes	Yes	Yes
16	$\Gamma_{16,3}$	Yes	Yes	Yes
16	$\Gamma_{16,4}$	Yes	Yes	Yes
16	$\Gamma_{16,13}$	Yes	Yes	Yes
17	$\mathbb{Z}_{17}$	Yes	Yes	Yes
18	$\Lambda_{18}[2]$	Yes	Yes	Yes
18	D_{18}	No	No	No
18	$S_3 \times \mathbb{Z}_3$	No	No	No
18	$\mathrm{Dih}(\mathbb{Z}_3 \times \mathbb{Z}_3)$	No	No	No
19	$\mathbb{Z}_{19}$	Yes	Yes	Yes
20	$\Lambda_{20}[2]$	Yes	Yes	Yes
20	D_{20}	No	No	No
20	Q_{20}	No	No	No
20	F_{20}	No	No	No
21	$\mathbb{Z}_{21}$	Yes	Yes	Yes
21	F_{21}	No	No	No
22	$\mathbb{Z}_{22}$	Yes	Yes	Yes
22	D_{22}	No	No	No
23	$\mathbb{Z}_{23}$	Yes	Yes	Yes

Table 1. The groups of order n with $1 \leq n \leq 23$.

Order	Group	CII group?	CCII group?	Is nilpotent of class at most 2?
24	$\Lambda_{24}[3]$	Yes	Yes	Yes
24	$\mathbb{Z}_4 \times D_6$	No	No	No
24	$\mathbb{Z}_2 \times Q_{12}$	No	No	No
24	$\mathbb{Z}_2 \times D_{12}$	No	No	No
24	$\mathbb{Z}_2 \times A_4$	No	No	No
24	$\mathbb{Z}_3 \times D_8$	Yes	Yes	Yes
24	D_{24}	No	No	No
24	S_4	No	No	No
24	Q_{24}	No	No	No
24	$SL(2,3)$	No	No	No
24	$\mathbb{Z}_3 \times Q_8$	Yes	Yes	Yes
24	$\Gamma_{24,1}$	No	No	No
24	$\Gamma_{24,8}$	No	No	No
25	$\Lambda_{25}[2]$	Yes	Yes	Yes
26	$\mathbb{Z}_{26}$	Yes	Yes	Yes
26	D_{26}	No	No	No
27	$\Lambda_{27}[3]$	Yes	Yes	Yes
27	M_{27}	Yes	Yes	Yes
27	$B(2,3)$	Yes	Yes	Yes
28	$\Lambda_{28}[2]$	Yes	Yes	Yes
28	D_{28}	No	No	No
28	Q_{28}	No	No	No
29	$\mathbb{Z}_{29}$	Yes	Yes	Yes
30	$\mathbb{Z}_{30}$	Yes	Yes	Yes
30	D_{30}	No	No	No
30	$\mathbb{Z}_3 \times D_{10}$	No	No	No
30	$\mathbb{Z}_5 \times S_3$	No	No	No
31	$\mathbb{Z}_{31}$	Yes	Yes	Yes

Table 2. The groups of order n with $24 \le n \le 31$.

Order	Group	CII group?	CCII group?	Is nilpotent of class at most 2?
32	$\Lambda_{32}[7]$	Yes	Yes	Yes
32	D_{32}	No	No	No
32	Q_{32}	No	No	No
32	$\Gamma_{32,49}$	Yes	Yes	Yes
32	SD_{32}	No	No	No
32	$\Gamma_{32,50}$	Yes	Yes	Yes
32	M_{32}	Yes	Yes	Yes
32	$\Gamma_{32,11}$	No	Yes	No
32	$\Gamma_{32,27}$	Yes	Yes	Yes
32	$\Gamma_{32,38}$	Yes	Yes	Yes
32	$\Gamma_{32,42}$	No	Yes	No
32	$\Gamma_{32,6}$	No	Yes	No
32	$\mathrm{Hol}(\mathbb{Z}_8)$	No	Yes	No
32	$\Gamma_{32,28}$	Yes	Yes	Yes
32	$\mathrm{Dih}(\mathbb{Z}_4 \times \mathbb{Z}_4)$	Yes	Yes	Yes
32	$\Gamma_{32,5}$	Yes	Yes	Yes
32	$\Gamma_{32,29}$	Yes	Yes	Yes
32	$\Gamma_{32,9}$	No	Yes	No
32	$\Gamma_{32,24}$	Yes	Yes	Yes
32	$\Gamma_{32,33}$	Yes	Yes	Yes
32	$\Gamma_{32,12}$	Yes	Yes	Yes
32	$\Gamma_{32,35}$	Yes	Yes	Yes
32	$\Gamma_{32,4}$	Yes	Yes	Yes
32	$\Gamma_{32,10}$	No	Yes	No
32	$\Gamma_{32,7}$	No	Yes	No
32	$\Gamma_{32,15}$	No	Yes	No
32	$\Gamma_{32,31}$	Yes	Yes	Yes
32	$\Gamma_{32,44}$	No	Yes	No
32	$\Gamma_{32,8}$	No	Yes	No
32	$\Gamma_{32,30}$	Yes	Yes	Yes
32	$\Gamma_{32,14}$	No	Yes	No
32	$\Gamma_{32,13}$	No	Yes	No
32	$\Gamma_{32,2}$	Yes	Yes	Yes
32	$\Gamma_{32,32}$	Yes	Yes	Yes
32	$\mathbb{Z}_4 \times D_8$	Yes	Yes	Yes
32	$\mathbb{Z}_2 \times D_{16}$	No	Yes	No
32	$\mathbb{Z}_2 \times SD_{16}$	No	Yes	No
32	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times D_8$	Yes	Yes	Yes
32	$\mathbb{Z}_2 \times M_{16}$	Yes	Yes	Yes
32	$\mathbb{Z}_4 \times Q_8$	Yes	Yes	Yes
32	$\mathbb{Z}_2 \times Q_{16}$	No	Yes	No
32	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times Q_8$	Yes	Yes	Yes
32	$\mathbb{Z}_2 \times \Gamma_{16,13}$	Yes	Yes	Yes
32	$\mathbb{Z}_2 \times \Gamma_{16,3}$	Yes	Yes	Yes
32	$\mathbb{Z}_2 \times \Gamma_{16,4}$	Yes	Yes	Yes

Table 3. The groups of order 32**.**

Order	Group	CII group?	CCII group?	Is nilpotent of class at most 2?
33	$\mathbb{Z}_{33}$	Yes	Yes	Yes
34	$\mathbb{Z}_{34}$	Yes	Yes	Yes
34	D_{34}	No	No	No
35	$\mathbb{Z}_{35}$	Yes	Yes	Yes
36	$\Lambda_{36}[4]$	Yes	Yes	Yes
36	D_{36}	No	No	No
36	Q_{36}	No	No	No
36	$\Gamma_{36,9}$	No	No	No
36	$\Gamma_{36,7}$	No	No	No
36	$\Gamma_{36,3}$	No	No	No
36	$S_3 \times S_3$	No	No	No
36	$\mathbb{Z}_6 \times S_3$	No	No	No
36	$\mathbb{Z}_3 \times A_4$	No	No	No
36	$\mathbb{Z}_3 \times Q_{12}$	No	No	No
36	$\mathbb{Z}_2 \times \mathrm{Dih}(\mathbb{Z}_3 \times \mathbb{Z}_3)$	No	No	No
37	$\mathbb{Z}_{37}$	Yes	Yes	Yes
38	$\mathbb{Z}_{38}$	Yes	Yes	Yes
38	D_{38}	No	No	No
39	$\mathbb{Z}_{39}$	Yes	Yes	Yes
39	$\mathbb{Z}_{13} \rtimes \mathbb{Z}_3$	No	No	No
40	$\Lambda_{40}[3]$	Yes	Yes	Yes
40	D_{40}	No	No	No
40	Q_{40}	No	No	No
40	$\Gamma_{40,8}$	No	No	No
40	$\Gamma_{40,3}$	No	No	No
40	$\Gamma_{40,1}$	No	No	No
40	$\mathbb{Z}_2 \times F_{20}$	No	No	No
40	$\mathbb{Z}_4 \times D_{10}$	No	No	No
40	$\mathbb{Z}_5 \times D_8$	Yes	Yes	Yes
40	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times D_{10}$	No	No	No
40	$\mathbb{Z}_5 \times Q_8$	Yes	Yes	Yes
40	$\mathbb{Z}_2 \times Q_{20}$	No	No	No
41	$\mathbb{Z}_{41}$	Yes	Yes	Yes
42	$\mathbb{Z}_{42}$	Yes	Yes	Yes
42	D_{42}	No	No	No
42	F_{42}	No	No	No
42	$\mathbb{Z}_7 \times S_3$	No	No	No
42	$\mathbb{Z}_3 \times D_{14}$	No	No	No
42	$\mathbb{Z}_2 \times F_{21}$	No	No	No
43	$\mathbb{Z}_{43}$	Yes	Yes	Yes
44	$\Lambda_{44}[2]$	Yes	Yes	Yes
44	D_{44}	No	No	No
44	Q_{44}	No	No	No
45	$\Lambda_{45}[2]$	Yes	Yes	Yes

Table 4. The groups of order n with $33 \leq n \leq 45$.

Order	Group	CII group?	CCII group?	Is nilpotent of class at most 2?
46	$\mathbb{Z}_{46}$	Yes	Yes	Yes
46	D_{46}	No	No	No
47	$\mathbb{Z}_{47}$	Yes	Yes	Yes
48	$\Lambda_{48}[5]$	Yes	Yes	Yes
48	D_{48}	No	No	No
48	Q_{48}	No	No	No
48	$GL(2,3)$	No	No	No
48	$\Gamma_{48,28}$	No	No	No
48	$\Gamma_{48,37}$	No	No	No
48	$\Gamma_{48,30}$	No	No	No
48	$\Gamma_{48,3}$	No	No	No
48	$\Gamma_{48,50}$	No	No	No
48	$\Gamma_{48,14}$	No	No	No
48	$\Gamma_{48,15}$	No	No	No
48	$\Gamma_{48,5}$	No	No	No
48	$\Gamma_{48,6}$	No	No	No
48	$\Gamma_{48,39}$	No	No	No
48	$\Gamma_{48,17}$	No	No	No
48	$\Gamma_{48,41}$	No	No	No
48	$\Gamma_{48,1}$	No	No	No
48	$\Gamma_{48,13}$	No	No	No
48	$\Gamma_{48,18}$	No	No	No
48	$\Gamma_{48,12}$	No	No	No
48	$\Gamma_{48,33}$	No	No	No
48	$\Gamma_{48,16}$	No	No	No
48	$\Gamma_{48,10}$	No	No	No
48	$\Gamma_{48,19}$	No	No	No
48	$\mathbb{Z}_2 \times S_4$	No	No	No
48	$\mathbb{Z}_4 \times A_4$	No	No	No
48	$S_3 \times D_8$	No	No	No
48	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times A_4$	No	No	No
48	$\mathbb{Z}_2 \times SL(2,3)$	No	No	No
48	$\mathbb{Z}_8 \times S_3$	No	No	No
48	$\mathbb{Z}_3 \times D_{16}$	No	Yes	No
48	$\mathbb{Z}_6 \times D_8$	Yes	Yes	Yes
48	$S_3 \times Q_8$	No	No	No
48	$\mathbb{Z}_2 \times D_{24}$	No	No	No
48	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$	No	No	No
48	$\mathbb{Z}_3 \times SD_{16}$	No	Yes	No
48	$\mathbb{Z}_3 \times M_{16}$	Yes	Yes	Yes
48	$\mathbb{Z}_6 \times Q_8$	Yes	Yes	Yes
48	$\mathbb{Z}_3 \times Q_{16}$	No	Yes	No
48	$\mathbb{Z}_4 \times Q_{12}$	No	No	No
48	$\mathbb{Z}_2 \times Q_{24}$	No	No	No
48	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times Q_{12}$	No	No	No
48	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times S_3$	No	No	No
48	$\mathbb{Z}_2 \times \Gamma_{24,8}$	No	No	No
48	$\mathbb{Z}_3 \times \Gamma_{16,13}$	Yes	Yes	Yes
48	$\mathbb{Z}_3 \times \Gamma_{16,3}$	Yes	Yes	Yes
48	$\mathbb{Z}_2 \times \Gamma_{24,1}$	No	No	No
48	$\mathbb{Z}_3 \times \Gamma_{16,4}$	Yes	Yes	Yes

Table 5. The groups of order n with $46 \leq n \leq 48$.

Order	Group	CII group?	CCII group?	Is nilpotent of class at most 2?
49	$\Lambda_{49}[2]$	Yes	Yes	Yes
50	$\Lambda_{50}[2]$	Yes	Yes	Yes
50	D_{50}	No	No	No
50	$\Gamma_{50,4}$	No	No	No
50	$\mathbb{Z}_5 \times D_{10}$	No	No	No
51	$\mathbb{Z}_{51}$	Yes	Yes	Yes
52	$\Lambda_{52}[2]$	Yes	Yes	Yes
52	D_{52}	No	No	No
52	Q_{52}	No	No	No
52	$\Gamma_{52,3}$	No	No	No
53	$\mathbb{Z}_{53}$	Yes	Yes	Yes
54	$\Lambda_{54}[3]$	Yes	Yes	Yes
54	D_{54}	No	No	No
54	$\mathrm{Hol}(\mathbb{Z}_9)$	No	No	No
54	$\Gamma_{54,5}$	No	No	No
54	$\Gamma_{54,8}$	No	No	No
54	$\Gamma_{54,7}$	No	No	No
54	$\Gamma_{54,14}$	No	No	No
54	$\mathbb{Z}_9 \times S_3$	No	No	No
54	$\mathbb{Z}_3 \times D_{18}$	No	No	No
54	$\mathbb{Z}_2 \times B(2,3)$	Yes	Yes	Yes
54	$\mathbb{Z}_3 \times \mathbb{Z}_3 \times S_3$	No	No	No
54	$\mathbb{Z}_2 \times M_{27}$	Yes	Yes	Yes
54	$\mathbb{Z}_3 \times \mathrm{Dih}(\mathbb{Z}_3 \times \mathbb{Z}_3)$	No	No	No
55	$\mathbb{Z}_{55}$	Yes	Yes	Yes
55	$\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$	No	No	No
56	$\Lambda_{56}[3]$	Yes	Yes	Yes
56	D_{56}	No	No	No
56	F_{56}	No	No	No
56	Q_{56}	No	No	No
56	$\Gamma_{56,7}$	No	No	No
56	$\Gamma_{56,1}$	No	No	No
56	$\mathbb{Z}_4 \times D_{14}$	No	No	No
56	$\mathbb{Z}_7 \times D_8$	Yes	Yes	Yes
56	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times D_{14}$	No	No	No
56	$\mathbb{Z}_7 \times Q_8$	Yes	Yes	Yes
56	$\mathbb{Z}_2 \times Q_{28}$	No	No	No
57	$\mathbb{Z}_{57}$	Yes	Yes	Yes
57	$\mathbb{Z}_{19} \rtimes \mathbb{Z}_3$	No	No	No
58	$\mathbb{Z}_{58}$	Yes	Yes	Yes
58	D_{58}	No	No	No
59	$\mathbb{Z}_{59}$	Yes	Yes	Yes

Table 6. The groups of order n with $49 \leq n \leq 59$.

Order	Group	CII group?	CCII group?	Is nilpotent of class at most 2?
60	$\Lambda_{60}[2]$	Yes	Yes	Yes
60	A_5	No	No	No
60	D_{60}	No	No	No
60	Q_{60}	No	No	No
60	$\Gamma_{60,7}$	No	No	No
60	$S_3 \times D_{10}$	No	No	No
60	$\mathbb{Z}_3 \times F_{20}$	No	No	No
60	$\mathbb{Z}_5 \times A_4$	No	No	No
60	$\mathbb{Z}_6 \times D_{10}$	No	No	No
60	$\mathbb{Z}_{10} \times S_3$	No	No	No
60	$\mathbb{Z}_5 \times Q_{12}$	No	No	No
60	$\mathbb{Z}_3 \times Q_{20}$	No	No	No
61	$\mathbb{Z}_{61}$	Yes	Yes	Yes
62	$\mathbb{Z}_{62}$	Yes	Yes	Yes
62	D_{62}	No	No	No
63	$\Lambda_{63}[2]$	Yes	Yes	Yes
63	$\Gamma_{63,1}$	No	No	No
63	$\mathbb{Z}_3 \times F_{21}$	No	No	No

Table 7. The groups of order n with $60 \leq n \leq 63$.