



On the average hitting times of weighted Cayley graphs*

Research Article

Yuuho Tanaka 

Abstract: In this paper, we give exact formulas for the average hitting times of random walks from one vertex to any other vertex on certain weighted Cayley graphs.

2020 MSC: 05C50 • 05C81

Keywords: Random walk • Hitting time • Weighted graph • Cayley graph • Kirchhoff index

1. Introduction

Let $G = (V, E)$ be a graph. A *weight* on G is a function $w : V \times V \rightarrow [0, +\infty)$ such that for $u, v \in V$,

$$\begin{cases} w((u, v)) > 0 & \text{if } (u, v) \in E, \\ w((u, v)) = 0 & \text{otherwise.} \end{cases}$$

In other words, w assigns a positive number $w((u, v))$ to each edge $(u, v) \in E$. Such a graph G is called a *weighted graph*. Every graph $G = (V, E)$ can be given the *trivial weight* $w((u, v)) = 1$ if $(u, v) \in E$ and $w((u, v)) = 0$ otherwise. In this case, we refer to G as an *unweighted graph*.

* This work was supported by JSPS KAKENHI Grant Numbers JP23KJ2020 and JP25K23339, and a Waseda Research Institute for Science and Engineering Grant-in-Aid for Young Scientists (Early Bird).

Yuuho Tanaka (Corresponding Author); Faculty of Science and Technology, Oita University, Oita, Oita 870-1192, Japan, (tanaka-yuuho@oita-u.ac.jp).

© 2026 The Author(s). Published by iPeak Academy Ltd. This is an open access article under the CC BY 4.0 license (<https://creativecommons.org/licenses/by/4.0/>).

A random walk on a graph G is a discrete stochastic model such that a random walker on a vertex $u \in V$ moves to an adjacent vertex v at the next step with probability $\frac{w((u,v))}{W(u)}$, where $W(u) = \sum_{x \in V} w((u,x))$. A random walk on an unweighted graph (namely $w((u,v))/W(u) = 1/\deg(u)$, where $\deg(u) = |\{v \in V \mid (u,v) \in E\}|$) is called a *simple random walk*. The number of steps required until the random walker starting from a vertex u of G will first arrive at a vertex v of G is called the *hitting time* from u to v of the random walk on G . The *average hitting time* (AHT) from u to v on G , which is denoted by $h(G; u, v)$, means the expected value of the hitting time from u to v of random walks on G .

Using an elementary method, Doi et al. [4] gave the exact formula for the AHTs of simple random walks on the unweighted Cayley graph $\text{Cay}(\mathbb{Z}_N, \{\pm 1, \pm 2\})$. Furthermore, Tanaka [10] gave the exact formula for the AHTs of simple random walks on the unweighted Cayley graph $\text{Cay}(\mathbb{Z}_N, \{+1, +2\})$. The purpose of the present paper is to derive exact formulas for the AHTs of random walks on certain weighted Cayley graphs by using matrix equations and matrix decompositions. Chang et al. [2] obtained an explicit formula for the AHTs of random walks on weighted undirected graphs by using the formula for the number of spanning trees of the graphs. Therefore, the aim of this paper is to obtain exact formulas for the AHTs of random walks on the weighted Cayley graphs $\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\})$ and $\text{Cay}(\mathbb{Z}_N, \{+1, +2\})$.

For undirected Cayley graphs, in the method using matrix equations, the entries of the inverse of the coefficient matrix can be expressed using the sequence of numbers appearing in the formula for the number of spanning trees [4, 8]. Therefore, we hypothesize that a similar property holds for weighted Cayley graphs. For a weighted graph, we can obtain the weighted spanning tree enumerator from the weighted matrix-tree theorem [6, 7]. Thus, we can deduce the entries of the inverse of the coefficient matrix in this case.

Our proof is concise and fully combinatorial. In particular, it does not require any spectral graph theoretical arguments. For the weighted directed Cayley graph $\text{Cay}(\mathbb{Z}_N, \{+1, +2\})$, we develop a novel method by combining the Laplacian matrix approach LU decomposition [5].

This paper is organized as follows. In Section 2, we present our main results concerning the exact formulas for the AHTs and discuss their implications. In Section 3, we provide fundamental definitions related to weighted graphs. In Section 4, we derive the exact formula for the AHTs of random walks on the weighted Cayley graph $\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\})$ in a special case, by decomposing the coefficient matrix of the matrix equations for the AHTs into a product of simple matrices. We also compute the inverse of each of these simple matrices to derive the exact formula. In Section 5, we present the exact formula for the AHTs of random walks on the weighted Cayley graph $\text{Cay}(\mathbb{Z}_N, \{+1, +2\})$ in a special case. Finally, in Section 6, we establish the exact formula for the Kirchhoff index of random walks on these weighted Cayley graphs.

2. Main results

In this section, we present our main results concerning the exact formulas for the AHTs on the weighted Cayley graphs.

Theorem 2.1. *For $1 \leq \ell \leq 2n - 1$, the exact formula for the AHTs of random walks on the weighted Cayley graph $\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\})$ is*

$$h(\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\}); 0, \ell) = \frac{1}{4p(1-p)} \begin{cases} \ell(2n - \ell) & \text{if } \ell \text{ is even,} \\ (\ell - 1)(2n - \ell + 1) + 4(1 - p)(n - \ell + p) & \text{if } \ell \text{ is odd,} \end{cases}$$

$$h(\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\}); 1, \ell + 1) = \frac{1}{4p(1-p)} \begin{cases} \ell(2n - \ell) & \text{if } \ell \text{ is even,} \\ (\ell - 1)(2n - \ell + 1) + 4p(n - \ell + 1 - p) & \text{if } \ell \text{ is odd.} \end{cases}$$

Theorem 2.2. *For $1 \leq \ell \leq N - 1$, an exact formula for the AHTs of random walks on the weighted Cayley graph $\text{Cay}(\mathbb{Z}_N, \{+1, +2\})$ is*

$$h(\text{Cay}(\mathbb{Z}_N, \{+1, +2\}); 0, \ell) = \frac{N(p-1)((p-1)^\ell - 1) - \ell((p-1)^N - 1)}{(p-2)((p-1)^N - 1)}.$$

These exact formulas under the weighted settings generalize the previous results for unweighted graphs provided by Doi et al. [4] and Tanaka [10]. Crucially, our formulas explicitly capture the dependence on the weight parameters p and q , revealing how the asymmetry of edge weights influences the expected random walk steps. As an important application of these main theorems, we can also evaluate the topological invariants of the underlying graphs. In Section 6, we utilize the above theorems to derive exact formulas for the Kirchhoff index of these weighted Cayley graphs, demonstrating the utility of our AHT formulas in electrical network theory.

3. Preliminaries

A directed graph (or digraph) is a couple $G = (V, E)$, where V is a set of *vertices* and E is a family of ordered pairs (x, y) of distinct elements in V , which is called the set of *edges*. If $(x, y) \in E$, then we say that vertices x and y are *incident* with the edge (x, y) *joining* x and y .

The *Laplacian matrix* $\text{Lap}_G = [\text{Lap}_G(i, j)]$ indexed by the vertices for a weighted graph G is defined as

$$\text{Lap}_G(i, j) = \begin{cases} \sum_{x \in V} w((i, x)) & \text{if } i = j, \\ -w((i, j)) & \text{otherwise.} \end{cases}$$

Let Lap'_G (resp. Lap''_G) be the matrix obtained from Lap_G by deleting the first (resp. last) row and column.

Let G be a finite group and let $S \subseteq G$ be a subset. The corresponding *Cayley graph* $\text{Cay}(G, S)$ has its vertex set equal to G . Two vertices $g, h \in G$ are joined by a directed edge from g to h if and only if there exists $s \in S$ such that $g = sh$. For any vertex x of a Cayley graph $\text{Cay}(G, S)$, the out-degree $\deg(x) = |S|$ is constant.

4. Weighted Cayley graph $\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\})$

We define a weight $w(\{v_i, v_j\})$ on the weighted undirected Cayley graph $\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\})$ as follows:

$$w(\{v_i, v_j\}) = \begin{cases} p & \text{if } j - i = 1, i \text{ is even,} \\ q & \text{if } j - i = 1, i \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases}$$

where $v_i = i + 2n\mathbb{Z} \in \mathbb{Z}_{2n}$, and for any adjacent vertices, the integer representatives $i, j \in \mathbb{Z}$ are chosen such that $j - i = 1$.

Let $\vec{h}$ be the column vector whose i -th entry is

$$\begin{cases} h_{2n}(1, 2n - i + 1) & \text{if } 1 \leq i \leq 2n - 1, i \text{ is odd,} \\ h_{2n}(0, 2n - i) & \text{if } 1 \leq i \leq 2n - 1, i \text{ is even,} \\ h_{2n}(1, 4n - i) & \text{if } 2n \leq i \leq 2(2n - 1), i \text{ is odd,} \\ h_{2n}(0, 4n - i - 1) & \text{if } 2n \leq i \leq 2(2n - 1), i \text{ is even,} \end{cases}$$

and let $\vec{1}$ be a column vector of proper dimensions whose entries are all 1.

On the weighted undirected Cayley graph $\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\})$, we can assume that a random walker starts from vertex 0 or 1 without loss of generality. By definition, in a random walk on $\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\})$, the walker moves with probability $\frac{p}{p+q}$ or $\frac{q}{p+q}$ to an arbitrary adjacent vertex. Moreover, by the symmetry

of $\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\})$, we have

$$\begin{aligned} h_{2n}(0, \ell) &= h_{2n}(2k, 2k + \ell) = h_{2n}(2k, 2k - 2n + \ell), \\ h_{2n}(1, \ell) &= h_{2n}(2k + 1, 2k + \ell) = h_{2n}(2k + 1, 2k - 2n + \ell) \end{aligned}$$

for all $k, \ell \in \mathbb{Z}_{2n}$. By the above properties, we have that $\forall \ell \in \mathbb{Z}_{2n} \setminus \{0\}$,

$$h_{2n}(0, \ell) = \frac{1}{p+q} (q(1 + h_{2n}(-1, \ell)) + p(1 + h_{2n}(1, \ell))),$$

and hence $\forall \ell \in \mathbb{Z}_{2n} \setminus \{0\}$,

$$(p+q)h_{2n}(0, \ell) - qh_{2n}(1, \ell+2) - ph_{2n}(1, \ell) = p+q.$$

Similarly, we have that $\forall \ell \in \mathbb{Z}_{2n} \setminus \{1\}$,

$$h_{2n}(1, \ell) = \frac{1}{p+q} (p(1 + h_{2n}(0, \ell)) + q(1 + h_{2n}(2, \ell))),$$

and hence $\forall \ell \in \mathbb{Z}_{2n} \setminus \{1\}$,

$$(p+q)h_{2n}(1, \ell) - ph_{2n}(0, \ell) - qh_{2n}(0, \ell-2) = p+q.$$

Therefore, we obtain

$$\begin{bmatrix} \text{Lap}'_{\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\})} & O \\ O & \text{Lap}''_{\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\})} \end{bmatrix} \vec{h} = (p+q)\vec{1}.$$

Let H_{2n} be the $2(2n-1) \times 2(2n-1)$ matrix whose (i, j) -th entry is

$$\begin{cases} \text{Lap}'_{\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\})}(i, j) & \text{if } 1 \leq i, j \leq 2n-1, \\ \text{Lap}''_{\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\})}(i-2n+1, j-2n+1) & \text{if } 2n \leq i, j \leq 2(2n-1), \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $H_{2n}\vec{h} = (p+q)\vec{1}$.

Our goal is to obtain an exact formula for the AHTs of a random walk on $\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\})$, so we need to solve the above matrix equation.

4.1. Proof of main theorem

To determine the AHTs, it suffices to solve the matrix equation $H_{2n}\vec{h} = (p+q)\vec{1}$. First, we decompose the coefficient matrix H_{2n} .

Assume that $p+q=1$. Let U_{2n} be the $2(2n-1) \times 2(2n-1)$ upper triangular matrix whose (i, j) -th entry is 1 if $j \geq i$ and 0 otherwise. Furthermore, we define the matrix R_{2n} as follows:

$$R_{2n}(i, j) := \begin{cases} 2 & \text{if } i \text{ is odd, } 1 \leq i = j < 2n, \\ 3-2p & \text{if } i \text{ is even, } 1 \leq i = j < 2n, \\ 2p & \text{if } i \text{ is odd, } 2n \leq i = j \leq 2(2n-1), \\ 2-p & \text{if } i \neq j, 1 \leq i, j < 2n, \\ p & \text{if } i \neq j, 2n < i \leq 2(2n-1), \\ & i \neq j, 2n < j \leq 2(2n-1), \\ 1 & \text{otherwise.} \end{cases}$$

Then the inverse matrix R_{2n}^{-1} is given by

$$R_{2n}^{-1}(i, j) = \frac{1}{n} \begin{cases} \frac{n-(1-p)}{p} & \text{if } i = j \text{ and } i \text{ is odd,} \\ \frac{n-p}{1-p} & \text{if } i = j \neq 2n \text{ and } i \text{ is even,} \\ \frac{2(n-p)}{1-p} & \text{if } i = j = 2n, \\ -\frac{1-p}{p} & \text{if } i \text{ and } j \text{ are odd, and either } i < 2n < j \text{ or } j < 2n < i, \\ -\frac{p}{1-p} & \text{if } i \text{ and } j \text{ are even, } i \neq j, \text{ and either } i \leq 2n \leq j \text{ or } j \leq 2n \leq i, \\ -1 & \text{if } i \not\equiv j \pmod{2}, \text{ and either } i \leq 2n \leq j \text{ or } j \leq 2n \leq i, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the following.

Theorem 4.1. $H_{2n} = U_{2n}^{-1} R_{2n} {}^t U_{2n}^{-1}$.

Proof. Let S_{2n} denote the matrix $U_{2n}^{-1} R_{2n} {}^t U_{2n}^{-1}$. By simple matrix computations, the entries of S_{2n} can be calculated as follows:

$$S_{2n}(i, j) = \begin{cases} R_{2n}(i, j) - R_{2n}(i, j+1) - R_{2n}(i+1, j) + R_{2n}(i+1, j+1) & \text{if } 1 \leq i, j < 2(2n-1), \\ R_{2n}(2(2n-1), j) - R_{2n}(2(2n-1), j+1) & \text{if } i = 2(2n-1), \\ & j \neq 2(2n-1), \\ R_{2n}(i, 2(2n-1)) - R_{2n}(i, 2(2n-1)) & \text{if } i \neq 2(2n-1), \\ & j = 2(2n-1), \\ R_{2n}(2(2n-1), 2(2n-1)) & \text{if } i = j = 2(2n-1), \end{cases}$$

$$= \begin{cases} 1 & \text{if } i = j, \\ -p & \text{if } i = j+1, i \text{ is odd,} \\ & i = j-1, i \text{ is even,} \\ -1+p & \text{if } i = j+1, i \neq 2n, i \text{ is even,} \\ & i = j-1, i \neq 2n-1, i \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have $S_{2n} = H_{2n}$. □

Defining a new variable vector $\vec{h}' = {}^t U_{2n}^{-1} \vec{h}$, the matrix equation $H_{2n} \vec{h} = \vec{1}$ can be expressed by a single matrix equation as

$$\vec{h}' = \begin{bmatrix} h'_1 \\ h'_2 \\ \vdots \\ h'_{2n-1} \\ h'_{2n} \\ h'_{2n+1} \\ \vdots \\ h'_{2(2n-1)-1} \\ h'_{2(2n-1)} \end{bmatrix} := \begin{bmatrix} h_{2n}(1, 2n) \\ h_{2n}(0, 2n-2) - h_{2n}(1, 2n) \\ \vdots \\ h_{2n}(1, 2) - h_{2n}(0, 2) \\ h_{2n}(0, 2n-1) - h_{2n}(1, 2) \\ h_{2n}(1, 2n-1) - h_{2n}(0, 2n-1) \\ \vdots \\ h_{2n}(1, 3) - h_{2n}(0, 3) \\ h_{2n}(0, 1) - h_{2n}(1, 3) \end{bmatrix} = R_{2n}^{-1} \begin{bmatrix} 2(2n-1) \\ 2(2n-1)-1 \\ \vdots \\ 2n \\ 2n-1 \\ 2n-2 \\ \vdots \\ 2 \\ 1 \end{bmatrix}.$$

Solving this matrix equation, we obtain the following lemma.

Lemma 4.2. For $1 \leq \ell \leq 2(2n-1)$,

$$h'_\ell = \begin{cases} \frac{n-\ell+p}{p} & \text{if } 1 \leq \ell \leq 2n-1, \ell \text{ is odd,} \\ \frac{n-\ell+p}{1-p} & \text{if } 1 \leq \ell \leq 2n-1, \ell \text{ is even,} \\ 0 & \text{if } \ell = 2n, \\ \frac{3n-\ell-p}{p} & \text{if } 2n+1 \leq \ell \leq 2(2n-1), \ell \text{ is odd,} \\ \frac{3n-\ell-p}{1-p} & \text{if } 2n+1 \leq \ell \leq 2(2n-1), \ell \text{ is even.} \end{cases}$$

Proof. For the case where $1 \leq \ell \leq 2n-1$ (ℓ is odd), we have

$$\begin{aligned} h'_\ell &= R_{2n}^{-1t} [4n-2, 4n-3, \dots, 1] \\ &= \sum_{k=1}^{2n-1} R_{2n}^{-1}(\ell, k)(4n-k-1) \\ &= \frac{n-(1-p)}{np}(4n-\ell-1) - \frac{1-p}{np} \sum_{\substack{1 \leq k \leq n \\ k \neq \frac{\ell+1}{2}}} 2(2n-k) - \frac{1}{n} \sum_{k=1}^n (2(2n-k)-1) \\ &= \frac{4n-\ell-1}{p} - \frac{1-p}{np} \sum_{k=1}^n 2(2n-k) - \frac{1}{n} \sum_{k=1}^n (2(2n-k)-1) \\ &= \frac{4n-\ell-1}{p} - \frac{1-p}{np}(4n^2-n(n+1)) - \frac{1}{n}(n(4n-1)-n(n+1)) \\ &= \frac{n-\ell+p}{p}. \end{aligned}$$

For the case where $1 \leq \ell \leq 2n-1$ (ℓ is even), we have

$$\begin{aligned} h'_\ell &= R_{2n}^{-1t} [4n-2, 4n-3, \dots, 1] \\ &= \sum_{k=1}^{2n-1} R_{2n}^{-1}(\ell, k)(4n-k-1) \\ &= \frac{n-p}{n(1-p)}(4n-\ell-1) - \frac{p}{n(1-p)} \sum_{\substack{1 \leq k \leq n \\ k \neq \frac{\ell}{2}}} (2(2n-k)-1) - \frac{1}{n} \sum_{k=1}^n 2(2n-k) \\ &= \frac{4n-\ell-1}{1-p} - \frac{p}{n(1-p)} \sum_{k=1}^n (2(2n-k)-1) - \frac{1}{n} \sum_{k=1}^n 2(2n-k) \\ &= \frac{4n-\ell-1}{1-p} - \frac{p}{n(1-p)}(n(4n-1)-n(n+1)) - \frac{1}{n}(4n^2-n(n+1)) \\ &= \frac{n-\ell+p}{1-p}. \end{aligned}$$

In the case of $\ell = 2n$, we have

$$\begin{aligned} h'_\ell &= R_{2n}^{-1t} [4n-2, 4n-3, \dots, 1] \\ &= \sum_{k=1}^{4n-3} R_{2n}^{-1}(2n, k)(4n-k-2) \\ &= \frac{2(n-p)}{n(1-p)}(2n-1) - \frac{p}{n(1-p)} \sum_{\substack{1 \leq k \leq 2n-1 \\ k \neq n}} (2(2n-k)-1) - \frac{1}{n} \sum_{k=1}^{2n-1} 2(2n-k) \end{aligned}$$

$$\begin{aligned}
&= \frac{(2n-1)(2n-p)}{n(1-p)} - \frac{p}{n(1-p)}((2n-1)(4n-1) - 2n(2n-1)) - \frac{1}{n}(4n(2n-1) - 2n(2n-1)) \\
&= \frac{(2n-1)(2n-p)}{n(1-p)} - \frac{(2n-1)^2 p}{n(1-p)} - 2(2n-1) \\
&= 0.
\end{aligned}$$

For the case where $2n+1 \leq \ell \leq 2(2n-1)$ (ℓ is odd), we have

$$\begin{aligned}
h'_\ell &= R_{2n}^{-1t} [4n-2, 4n-3, \dots, 1] \\
&= \sum_{k=2n-1}^{4n-3} R_{2n}^{-1}(\ell, k)(4n-k-1) \\
&= \frac{n-(1-p)}{np}(4n-\ell-1) - \frac{1-p}{np} \sum_{\substack{n+1 \leq k \leq 2n-1 \\ k \neq \frac{\ell+1}{2}}} 2(2n-k) - \frac{1}{n} \sum_{k=n}^{2n-1} (2(2n-k)-1) \\
&= \frac{4n-\ell-1}{p} - \frac{1-p}{np} \sum_{k=n+1}^{2n-1} 2(2n-k) - \frac{1}{n} \sum_{k=n}^{2n-1} (2(2n-k)-1) \\
&= \frac{4n-\ell-1}{p} - \frac{1-p}{np} \sum_{k=1}^{n-1} 2(n-k) - \frac{1}{n} \sum_{k=0}^{n-1} (2(n-k)-1) \\
&= \frac{4n-\ell-1}{p} - \frac{1-p}{np}(2n(n-1) - n(n-1)) - \frac{1}{n}(n(2n-1) - n(n-1)) \\
&= \frac{3n-\ell-p}{p}.
\end{aligned}$$

For the case where $2n+1 \leq \ell \leq 2(2n-1)$ (ℓ is even), we have

$$\begin{aligned}
h'_\ell &= R_{2n}^{-1t} [4n-2, 4n-3, \dots, 1] \\
&= \sum_{k=2n-1}^{4n-3} R_{2n}^{-1}(\ell, k)(4n-k-1) \\
&= \frac{n-p}{n(1-p)}(4n-\ell-1) - \frac{p}{n(1-p)} \sum_{\substack{n \leq k \leq 2n-1 \\ k \neq \frac{\ell}{2}}} (2(2n-k)-1) - \frac{1}{n} \sum_{k=n+1}^{2n-1} 2(2n-k) \\
&= \frac{4n-\ell-1}{1-p} - \frac{p}{n(1-p)} \sum_{k=n}^{2n-1} (2(2n-k)-1) - \frac{1}{n} \sum_{k=n+1}^{2n-1} 2(2n-k) \\
&= \frac{4n-\ell-1}{1-p} - \frac{p}{n(1-p)} \sum_{k=0}^{n-1} (2(n-k)-1) - \frac{1}{n} \sum_{k=1}^{n-1} 2(n-k) \\
&= \frac{4n-\ell-1}{1-p} - \frac{p}{n(1-p)}(n(2n-1) - n(n-1)) - \frac{1}{n}(2n(n-1) - n(n-1)) \\
&= \frac{3n-\ell-p}{1-p}.
\end{aligned}$$

□

Combining [Lemma 4.2](#) and the equations

$$h_{0,\ell} = \begin{cases} \sum_{i=1}^{2n-\ell} h'_i & \text{if } \ell \text{ is even,} \\ \sum_{i=1}^{4n-\ell-1} h'_i & \text{if } \ell \text{ is odd,} \end{cases} \text{ and } h_{1,\ell+1} = \begin{cases} \sum_{i=1}^{4n-\ell-1} h'_i & \text{if } \ell \text{ is even,} \\ \sum_{i=1}^{2n-\ell} h'_i & \text{if } \ell \text{ is odd,} \end{cases} \text{ for } 1 \leq \ell \leq 2n-1,$$

we obtain [Theorem 2.1](#).

Proof. For the case where ℓ is even, we have

$$\begin{aligned}
 h_{2n}(0, \ell) &= \sum_{i=1}^{\frac{2n-\ell}{2}} \frac{n - (2i-1) + p}{p} + \sum_{i=1}^{\frac{2n-\ell}{2}} \frac{n - 2i + p}{1-p} \\
 &= \sum_{i=1}^{\frac{2n-\ell}{2}} \frac{1}{p} + \sum_{i=1}^{\frac{2n-\ell}{2}} \frac{n - 2i + p}{p(1-p)} \\
 &= \frac{2n-\ell}{2p} + \frac{(2n-\ell)(n+p)}{2p(1-p)} - \frac{(2n-\ell)(2n-\ell+2)}{4p(1-p)} \\
 &= \frac{\ell(2n-\ell)}{4p(1-p)}.
 \end{aligned}$$

For the case where ℓ is odd ($\ell \neq 1, 2n-1$), we have

$$\begin{aligned}
 h_{2n}(0, \ell) &= \sum_{i=1}^n \frac{n - (2i-1) + p}{p} + \sum_{i=1}^{n-1} \frac{n - 2i + p}{1-p} + \sum_{i=n+1}^{\frac{4n-\ell-1}{2}} \frac{3n - (2i-1) - p}{p} + \sum_{i=n+1}^{\frac{4n-\ell-1}{2}} \frac{3n - 2i - p}{1-p} \\
 &= \sum_{i=1}^n \frac{n - (2i-1) + p}{p} + \sum_{i=1}^{n-1} \frac{n - 2i + p}{1-p} + \sum_{i=1}^{\frac{2n-\ell-1}{2}} \frac{n - (2i-1) - p}{p} + \sum_{i=1}^{\frac{2n-\ell-1}{2}} \frac{n - 2i - p}{1-p} \\
 &= \sum_{i=1}^n \frac{1}{p} - \frac{n-p}{p} + \sum_{i=1}^{n-1} \frac{n - 2i + p}{p(1-p)} + \sum_{i=1}^{\frac{2n-\ell-1}{2}} \frac{1}{p} + \sum_{i=1}^{\frac{2n-\ell-1}{2}} \frac{n - 2i - p}{p(1-p)} \\
 &= \frac{n}{p} - \frac{n-p}{p} + \frac{(n-1)(n+p)}{p(1-p)} - \frac{n(n-1)}{p(1-p)} \\
 &\quad + \frac{2n-\ell-1}{2p} + \frac{(2n-\ell-1)(n-p)}{2p(1-p)} - \frac{(2n-\ell-1)(2n-\ell+1)}{4p(1-p)} \\
 &= \frac{2n+2p-\ell-1}{2p} + \frac{p(n-1)}{p(1-p)} + \frac{(2n-\ell-1)(-2p+\ell-1)}{4p(1-p)} \\
 &= \frac{(\ell-1)(2n-\ell+1) + 4(1-p)(n-\ell+p)}{4p(1-p)}.
 \end{aligned}$$

For the case where $\ell = 2n-1$, we have

$$\begin{aligned}
 h_{2n}(0, 2n-1) &= \sum_{i=1}^n \frac{n - (2i-1) + p}{p} + \sum_{i=1}^{n-1} \frac{n - 2i + p}{1-p} \\
 &= \sum_{i=1}^n \frac{1}{p} - \frac{n-p}{p} + \sum_{i=1}^{n-1} \frac{n - 2i + p}{p(1-p)} \\
 &= \frac{n}{p} - \frac{n-p}{p} + \frac{(n+p)(n-1)}{p(1-p)} - \frac{n(n-1)}{p(1-p)} \\
 &= \frac{(n-1) - (1-p)(n-p-1)}{p(1-p)} \\
 &= \frac{(2n-1-1)(2n-(2n-1)+1) + 4(1-p)(n-(2n-1)+p)}{4p(1-p)}.
 \end{aligned}$$

For the case where $\ell = 1$, we have

$$\begin{aligned}
 h_{2n}(0, 1) &= \sum_{i=1}^n \frac{n - (2i - 1) + p}{p} + \sum_{i=1}^{n-1} \frac{n - 2i + p}{1 - p} + \sum_{i=n+1}^{2n-1} \frac{3n - (2i - 1) - p}{p} + \sum_{i=n+1}^{2n-1} \frac{3n - 2i - p}{1 - p} \\
 &= \sum_{i=1}^n \frac{n - (2i - 1) + p}{p} + \sum_{i=1}^{n-1} \frac{n - 2i + p}{1 - p} + \sum_{i=1}^{n-1} \frac{n - (2i - 1) - p}{p} + \sum_{i=1}^{n-1} \frac{n - 2i - p}{1 - p} \\
 &= \sum_{i=1}^n \frac{1}{p} - \frac{n - p}{p} + \sum_{i=1}^{n-1} \frac{n - 2i + p}{p(1 - p)} + \sum_{i=1}^{n-1} \frac{1}{p} + \sum_{i=1}^{n-1} \frac{n - 2i - p}{p(1 - p)} \\
 &= \frac{n}{p} - \frac{n - p}{p} + \frac{2n(n - 1)}{p(1 - p)} - \frac{2n(n - 1)}{p(1 - p)} + \frac{n - 1}{p} \\
 &= \frac{n - 1 + p}{p} \\
 &= \frac{(1 - 1)(2n - 1 + 1) + 4(1 - p)(n - 1 + p)}{4p(1 - p)}.
 \end{aligned}$$

For the case where ℓ is even, we have

$$\begin{aligned}
 h_{2n}(1, \ell + 1) &= \sum_{i=1}^n \frac{n - (2i - 1) + p}{p} + \sum_{i=1}^{n-1} \frac{n - 2i + p}{1 - p} + \sum_{i=n+1}^{\frac{4n-\ell}{2}} \frac{3n - (2i - 1) - p}{p} + \sum_{i=n+1}^{\frac{4n-\ell-2}{2}} \frac{3n - 2i - p}{1 - p} \\
 &= \sum_{i=1}^n \frac{n - (2i - 1) + p}{p} + \sum_{i=1}^{n-1} \frac{n - 2i + p}{1 - p} + \sum_{i=1}^{\frac{2n-\ell}{2}} \frac{n - (2i - 1) - p}{p} + \sum_{i=1}^{\frac{2n-\ell-2}{2}} \frac{n - 2i - p}{1 - p} \\
 &= \sum_{i=1}^n \frac{1}{p} - \frac{n - p}{p} + \sum_{i=1}^{n-1} \frac{n - 2i + p}{p(1 - p)} + \sum_{i=1}^{\frac{2n-\ell}{2}} \frac{1}{p} + \frac{n - (2n - \ell) - p}{p} + \sum_{i=1}^{\frac{2n-\ell-2}{2}} \frac{n - 2i - p}{p(1 - p)} \\
 &= \frac{n}{p} - \frac{n - p}{p} + \frac{(n - 1)(n + p)}{p(1 - p)} - \frac{n(n - 1)}{p(1 - p)} + \frac{2n - \ell}{2p} - \frac{n - \ell + p}{p} \\
 &\quad + \frac{(2n - \ell - 2)(n - p)}{2p(1 - p)} - \frac{(2n - \ell - 2)(2n - \ell)}{4p(1 - p)} \\
 &= \frac{p(n - p)}{p(1 - p)} + \frac{\ell - 2p}{2p} + \frac{(2n - \ell - 2)(\ell - 2p)}{4p(1 - p)} \\
 &= \frac{\ell(2n - \ell)}{4p(1 - p)}.
 \end{aligned}$$

For the case where ℓ is odd, we have

$$\begin{aligned}
 h_{2n}(1, \ell + 1) &= \sum_{i=1}^{\frac{2n-\ell+1}{2}} \frac{n - (2i - 1) + p}{p} + \sum_{i=1}^{\frac{2n-\ell-1}{2}} \frac{n - 2i + p}{1 - p} \\
 &= \sum_{i=1}^{\frac{2n-\ell+1}{2}} \frac{1}{p} - \frac{n - \ell - p + 1}{p} + \sum_{i=1}^{\frac{2n-\ell-1}{2}} \frac{n - 2i + p}{p(1 - p)} \\
 &= \frac{2n - \ell + 1}{2p} - \frac{n - \ell - p + 1}{p} + \frac{(2n - \ell - 1)(n + p)}{2p(1 - p)} - \frac{(2n - \ell - 1)(2n - \ell + 1)}{4p(1 - p)} \\
 &= \frac{\ell - 1 + 2p}{2p} + \frac{(2n - \ell - 1)(\ell - 1) + 2p(2n - \ell - 1)}{4p(1 - p)} \\
 &= \frac{(\ell - 1)(2n - \ell + 1) + 4p(n - \ell + 1 - p)}{4p(1 - p)}.
 \end{aligned}$$

□

From Theorem 4.1, we have $H_{2n}^{-1} = {}^tU_{2n} R_{2n}^{-1} U_{2n}$. Hence, the entries of H_{2n}^{-1} can also be calculated as follows. For $1 \leq i, j \leq 2(2n-1)$, we have

$$H_{2n}^{-1}(i, j) = \frac{1}{4np(1-p)} \times \begin{cases} (2n-i-1+2p)(j+1-2p) & \text{if } 1 \leq j \leq i \leq 2n-1, i \text{ is odd, } j \text{ is odd,} \\ (2n-i)(j+1-2p) & \text{if } 1 \leq j \leq i \leq 2n-1, i \text{ is even, } j \text{ is odd,} \\ (2n-i-1+2p)j & \text{if } 1 \leq j \leq i \leq 2n-1, i \text{ is odd, } j \text{ is even,} \\ (2n-i)j & \text{if } 1 \leq j \leq i \leq 2n-1, i \text{ is even, } j \text{ is even,} \\ (2n-j-1+2p)(i+1-2p) & \text{if } 1 \leq i < j \leq 2n-1, i \text{ is odd, } j \text{ is odd,} \\ (2n-j-1+2p)i & \text{if } 1 \leq i < j \leq 2n-1, i \text{ is even, } j \text{ is odd,} \\ (2n-j)(i+1-2p) & \text{if } 1 \leq i < j \leq 2n-1, i \text{ is odd, } j \text{ is even,} \\ (2n-j)i & \text{if } 1 \leq i < j \leq 2n-1, i \text{ is even, } j \text{ is even,} \\ (2n-4n+i+2p)(4n-j-2p) & \text{if } 2n \leq j \leq i \leq 2(2n-1), i \text{ is odd, } j \text{ is odd,} \\ (2n-4n+i+1)(4n-j-2p) & \text{if } 2n \leq j \leq i \leq 2(2n-1), i \text{ is even, } j \text{ is odd,} \\ (2n-4n+i+2p)(4n-j-1) & \text{if } 2n \leq j \leq i \leq 2(2n-1), i \text{ is odd, } j \text{ is even,} \\ (2n-4n+i+1)(4n-j-1) & \text{if } 2n \leq j \leq i \leq 2(2n-1), i \text{ is even, } j \text{ is even,} \\ (2n-4n+j+2p)(4n-i-2p) & \text{if } 2n \leq i < j \leq 2(2n-1), i \text{ is odd, } j \text{ is odd,} \\ (2n-4n+j+2p)(4n-i-1) & \text{if } 2n \leq i < j \leq 2(2n-1), i \text{ is even, } j \text{ is odd,} \\ (2n-4n+j+1)(4n-i-2p) & \text{if } 2n \leq i < j \leq 2(2n-1), i \text{ is odd, } j \text{ is even,} \\ (2n-4n+j+1)(4n-i-1) & \text{if } 2n \leq i < j \leq 2(2n-1), i \text{ is even, } j \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

5. Weighted Cayley graph $\text{Cay}(\mathbb{Z}_N, \{+1, +2\})$

We define a weight $w((v_i, v_j))$ on the weighted directed Cayley graph $\text{Cay}(\mathbb{Z}_N, \{+1, +2\})$ as follows:

$$w((v_i, v_j)) = \begin{cases} p & \text{if } j-i=1, \\ q & \text{if } j-i=2, \\ 0 & \text{otherwise,} \end{cases}$$

where $i, j \in \mathbb{Z}$ and $v_i = i + N\mathbb{Z} \in \mathbb{Z}_N$.

Let $h_N(k, \ell)$ denote the AHT from a vertex v_k to a vertex v_ℓ on the weighted directed Cayley graph $\text{Cay}(\mathbb{Z}_N, \{+1, +2\})$. Let $\vec{h}$ be the $(N-1)$ -dimensional column vector whose i -th entry is $h_N(0, i)$ for $1 \leq i \leq N-1$, and let $\vec{1}$ be a column vector of proper dimensions whose entries are all 1.

On the weighted directed Cayley graph $\text{Cay}(\mathbb{Z}_N, \{+1, +2\})$, we can assume that a random walker starts from vertex 0 without loss of generality. By definition, the walker moves with probability $\frac{p}{p+q}$ or $\frac{q}{p+q}$ to an arbitrary vertex adjacent to the current vertex. Moreover, by the symmetry of $\text{Cay}(\mathbb{Z}_N, \{+1, +2\})$, we have

$$h_N(0, \ell) = h_N(k, k+\ell) = h_N(k+N-\ell, k) \quad \text{for all } k, \ell \in \mathbb{Z}_N.$$

Note that $h_N(0, 0) = 0$. By the above properties, for any $\ell \in \mathbb{Z}_N \setminus \{0\}$, we have

$$h_N(0, \ell) = \frac{p}{p+q}(1 + h_N(1, \ell)) + \frac{q}{p+q}(1 + h_N(2, \ell)),$$

and hence $\forall \ell \in \mathbb{Z}_N \setminus \{0\}$,

$$(p+q)h_N(0, \ell) - ph_N(0, \ell-1) - qh_N(0, \ell-2) = p+q.$$

We thus have ${}^t\text{Lap}''_{\text{Cay}(\mathbb{Z}_N, \{+1, +2\})} \vec{h} = (p+q)\vec{1}$. Let H_N be the $(N-1) \times (N-1)$ matrix whose (i, j) -th entry is ${}^t\text{Lap}''_{\text{Cay}(\mathbb{Z}_N, \{+1, +2\})}(i, j)$.

Then we obtain the matrix equation $H_N \vec{h} = (p+q)\vec{1}$.

Our goal is to obtain an exact formula for the AHTs of random walks on $\text{Cay}(\mathbb{Z}_N, \{+1, +2\})$, so we need to solve the above matrix equation.

5.1. Proof of main theorem

To determine the AHTs, it suffices to solve the matrix equation $H_N \vec{h} = (p+q)\vec{1}$. First, we utilize LU decomposition to analyze the coefficient matrix H_N .

Definition 5.1 (*LU decomposition*). Let M_N denote the set of all $N \times N$ matrices over $\mathbb{C}$. For $A \in M_N$, a factorization $A = LU$, where $L \in M_N$ is a lower triangular matrix and $U \in M_N$ is an upper triangular matrix, is called an *LU decomposition* of A .

Let $p+q=1$, and let P_N be the matrix

$$P_N := \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let L_N be the matrix

$$L_N := \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{p} & 1 & 0 & 0 & 0 & \cdots & 0 \\ -\frac{1+p}{p} & -1+p-p^2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & p(-1+p) & -p & 1 & 0 & \ddots & \vdots \\ 0 & 0 & -1+p & -p & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & -1+p & -p & 1 \end{bmatrix}.$$

Then, its inverse L_N^{-1} is

$$L_N^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{p} & 1 & 0 & 0 & 0 & \cdots & 0 \\ \frac{(p-1)^2-1}{p-2} & \frac{(p-1)^3-1}{p-2} & 1 & 0 & 0 & \cdots & 0 \\ \frac{(p-1)^3-1}{p-2} & \frac{(p-1)^4-1}{p-2} & \frac{(p-1)^2-1}{p-2} & 1 & 0 & \ddots & \vdots \\ \frac{(p-1)^4-1}{p-2} & \frac{(p-1)^5-1}{p-2} & \frac{(p-1)^3-1}{p-2} & \frac{(p-1)^2-1}{p-2} & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \frac{(p-1)^{N-2}-1}{p-2} & \frac{(p-1)^{N-1}-1}{p-2} & \frac{(p-1)^{N-3}-1}{p-2} & \cdots & \frac{(p-1)^3-1}{p-2} & \frac{(p-1)^2-1}{p-2} & 1 \end{bmatrix}.$$

Let U_N be the matrix

$$U_N := \begin{bmatrix} -p & 1 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{p} & 0 & \cdots & 0 & -1+p \\ 0 & 0 & 1 & \ddots & \vdots & \frac{(p-1)^4-p+1}{p-2} \\ 0 & 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & \frac{(p-1)^N-p+1}{p-2} \\ 0 & 0 & \cdots & 0 & 0 & \frac{(p-1)^N-1}{p-2} \end{bmatrix}.$$

Then, its inverse U_N^{-1} is

$$U_N^{-1} = \begin{bmatrix} -\frac{1}{p} & 1 & 0 & \cdots & 0 & \frac{(p-1)^2-p+1}{1-(p-1)^N} \\ 0 & p & 0 & \cdots & 0 & \frac{(p-1)^3-p+1}{1-(p-1)^N} \\ 0 & 0 & 1 & \ddots & \vdots & \frac{(p-1)^4-p+1}{1-(p-1)^N} \\ 0 & 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & \frac{(p-1)^N-p+1}{1-(p-1)^N} \\ 0 & 0 & \cdots & 0 & 0 & \frac{p-2}{(p-1)^N-1} \end{bmatrix}.$$

Then we have the following.

Theorem 5.2. $H_N = P_N L_N U_N$.

Proof. Let S_N denote the matrix $P_N L_N U_N$. By simple matrix computation, the entries of S_N can be calculated as

$$S_N(i, j) = \begin{cases} 1 & \text{if } i = j \neq N-1, \\ -p & \text{if } i = j+1, 4 \leq i \leq N-1, \\ -1+p & \text{if } i = j+2, 3 \leq i \leq N-1, \\ & (i, j) = (1, N-1), \\ -\frac{1}{p} + \frac{1}{p} & \text{if } (i, j) = (1, 2), \\ -\frac{-1+p}{p} + \frac{-1+p-p^2}{p} & \text{if } (i, j) = (3, 2), \\ (-1+p)(-1+p-p^2) + \frac{(p-1)^4-p+1}{p-2} & \text{if } (i, j) = (3, N-1), \\ p(-1+p)^2 - \frac{p((p-1)^4-p+1)}{p-2} + \frac{(p-1)^5-p+1}{p-2} & \text{if } (i, j) = (4, N-1), \\ (-1+p)\frac{(p-1)^{i-1}-p+1}{p-2} - p\frac{(p-1)^i-p+1}{p-2} + \frac{(p-1)^{i+1}-p+1}{p-2} & \text{if } j = N-1, 5 \leq i \leq N-2, \\ (-1+p)\frac{(p-1)^{N-2}-p+1}{p-2} - p\frac{(p-1)^{N-1}-p+1}{p-2} + \frac{(p-1)^N-1}{p-2} & \text{if } i = j = N-1, \\ 0 & \text{otherwise.} \end{cases}$$

For the case where $(i, j) = (1, 2)$, we have

$$S_N(i, j) = -\frac{1}{p} + \frac{1}{p} = 0.$$

For the case where $(i, j) = (3, 2)$, we have

$$S_N(i, j) = -\frac{-1+p}{p} + \frac{-1+p-p^2}{p} = -p.$$

For the case where $(i, j) = (3, N - 1)$, we have

$$\begin{aligned}
 S_N(i, j) &= (-1 + p)(-1 + p - p^2) + \frac{(p - 1)^4 - p + 1}{p - 2} \\
 &= \frac{(p - 1)^2(p - 2) - p^2(p - 1)(p - 2)}{p - 2} + \frac{(p - 1)^4 - p + 1}{p - 2} \\
 &= \frac{(p - 1)(p^2 - 3p + 2 - (p^3 - 2p^2)) + (p^3 - 3p^2 + 3p - 1) - 1}{p - 2} \\
 &= 0.
 \end{aligned}$$

For the case where $(i, j) = (4, N - 1)$, we have

$$\begin{aligned}
 S_N(i, j) &= p(-1 + p)^2 - \frac{p((p - 1)^4 - p + 1)}{p - 2} + \frac{(p - 1)^5 - p + 1}{p - 2} \\
 &= \frac{p(p - 1)^2(p - 2)}{p - 2} + \frac{-p(p - 1)^4 + p(p - 1) + (p - 1)^5 - (p - 1)}{p - 2} \\
 &= \frac{p(p - 1)^2(p - 2) - (p - 1)^4 + (p - 1)^2}{p - 2} \\
 &= 0.
 \end{aligned}$$

For the case where $j = N - 1$, $5 \leq i \leq N - 2$, we have

$$\begin{aligned}
 S_N(i, j) &= (-1 + p) \frac{(p - 1)^{i-1} - p + 1}{p - 2} - p \frac{(p - 1)^i - p + 1}{p - 2} + \frac{(p - 1)^{i+1} - p + 1}{p - 2} \\
 &= \frac{(p - 1)^i - (p - 1)^2 - p(p - 1)^i + p(p - 1) + (p - 1)^{i+1} - (p - 1)}{p - 2} \\
 &= \frac{-(p - 1)^{i+1} - (p - 1)^2 + (p - 1)^2 + (p - 1)^{i+1}}{p - 2} \\
 &= 0.
 \end{aligned}$$

For the case where $i = j = N - 1$, we have

$$\begin{aligned}
 S_N(i, j) &= (-1 + p) \frac{(p - 1)^{N-2} - p + 1}{p - 2} - p \frac{(p - 1)^{N-1} - p + 1}{p - 2} + \frac{(p - 1)^N - 1}{p - 2} \\
 &= \frac{(p - 1)^{N-1} - (p - 1)^2 - p(p - 1)^{N-1} + p(p - 1) + (p - 1)^N - 1}{p - 2} \\
 &= \frac{-(p - 1)^N - (p^2 - 2p + 1) + p^2 - p + (p - 1)^N - 1}{p - 2} \\
 &= 1.
 \end{aligned}$$

Therefore, we have $S_N = H_N$. □

From [Theorem 5.2](#), the matrix equation $H_N \vec{h} = \vec{1}$ can be expressed by the single matrix equation

$$\vec{h} = \begin{bmatrix} h_N(0, 1) \\ h_N(0, 2) \\ \vdots \\ h_N(0, \ell) \\ \vdots \\ h_N(0, N - 1) \end{bmatrix} = U_N^{-1} L_N^{-1} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Solving this matrix equation, we obtain [Theorem 2.2](#).

Proof. For the case where $\ell = 1$, we have

$$\begin{aligned}
 h_N(0, 1) &= 1 + \frac{(p-1)^2 - p + 1}{1 - (p-1)^N} \sum_{i=1}^{N-1} \frac{(p-1)^i - 1}{p-2} \\
 &= 1 + \frac{(p-1)^2 - p + 1}{(p-2)(1 - (p-1)^N)} \left(\frac{(p-1)(1 - (p-1)^{N-1})}{1 - (p-1)} - N + 1 \right) \\
 &= 1 - \frac{p-1}{(p-1)^N - 1} \left(\frac{(p-1)^N - 1}{p-2} - N \right) \\
 &= \frac{(p-2)((p-1)^N - 1) - (p-1)((p-1)^N - 1) + N(p-1)(p-2)}{(p-2)((p-1)^N - 1)} \\
 &= \frac{N(p-1)((p-1)^1 - 1) - ((p-1)^N - 1)}{(p-2)((p-1)^N - 1)}.
 \end{aligned}$$

For the case where $\ell = 2$, we have

$$\begin{aligned}
 h_N(0, 2) &= 1 + p + \frac{(p-1)^3 - p + 1}{1 - (p-1)^N} \sum_{i=1}^{N-1} \frac{(p-1)^i - 1}{p-2} \\
 &= 1 + p + \frac{(p-1)^3 - p + 1}{(p-2)(1 - (p-1)^N)} \left(\frac{(p-1)(1 - (p-1)^{N-1})}{1 - (p-1)} - N + 1 \right) \\
 &= 1 + p - \frac{(p-1)((p-1)^2 - 1)}{(p-2)((p-1)^N - 1)} \left(\frac{(p-1)^N - 1}{p-2} - N \right) \\
 &= \frac{(p+1)(p-2)^2}{(p-2)^2} - \frac{(p-1)((p-1)^2 - 1)}{(p-2)^2} + \frac{N(p-1)((p-1)^2 - 1)}{(p-2)((p-1)^N - 1)} \\
 &= \frac{(p+1)(p-2)^2}{(p-2)^2} - \frac{p(p-1)(p-2)}{(p-2)^2} + \frac{N(p-1)((p-1)^2 - 1)}{(p-2)((p-1)^N - 1)} \\
 &= \frac{(p+1)(p-2) - p(p-1)}{(p-2)} + \frac{N(p-1)((p-1)^2 - 1)}{(p-2)((p-1)^N - 1)} \\
 &= \frac{N(p-1)((p-1)^2 - 1) - 2((p-1)^N - 1)}{(p-2)((p-1)^N - 1)}.
 \end{aligned}$$

For the case where $3 \leq \ell \leq N-2$, we have

$$\begin{aligned}
 h_N(0, \ell) &= \sum_{i=1}^{\ell} \frac{(p-1)^i - 1}{p-2} + \frac{(p-1)^{\ell+1} - p + 1}{1 - (p-1)^N} \sum_{i=1}^{N-1} \frac{(p-1)^i - 1}{p-2} \\
 &= \frac{1}{p-2} \left(\sum_{i=1}^{\ell} (p-1)^i - \ell \right) + \frac{(p-1)^{\ell+1} - p + 1}{(p-2)(1 - (p-1)^N)} \left(\sum_{i=1}^{N-1} (p-1)^i - N + 1 \right) \\
 &= \frac{1}{p-2} \left(\frac{(p-1)(1 - (p-1)^{\ell})}{1 - (p-1)} - \ell + \frac{(p-1)^{\ell+1} - p + 1}{1 - (p-1)^N} \left(\frac{(p-1)(1 - (p-1)^{N-1})}{1 - (p-1)} - N + 1 \right) \right) \\
 &= \frac{1}{p-2} \left(\frac{(p-1)^{\ell+1} - p + 1}{p-2} - \ell - \frac{(p-1)^{\ell+1} - p + 1}{(p-1)^N - 1} \left(\frac{(p-1)^N - p + 1}{p-2} - N + 1 \right) \right) \\
 &= \frac{1}{p-2} \left(\frac{(p-1)^{\ell+1} - p + 1}{p-2} - \ell - \frac{(p-1)^{\ell+1} - p + 1}{(p-1)^N - 1} \left(\frac{(p-1)^N - 1}{p-2} - N \right) \right) \\
 &= \frac{N(p-1)((p-1)^{\ell} - 1) - \ell((p-1)^N - 1)}{(p-2)((p-1)^N - 1)}.
 \end{aligned}$$

For the case where $\ell = N - 1$, we have

$$\begin{aligned}
 h_N(0, N-1) &= \frac{p-2}{(p-1)^N - 1} \sum_{i=1}^{N-1} \frac{(p-1)^i - 1}{p-2} \\
 &= \frac{1}{(p-1)^N - 1} \left(\frac{(p-1)(1 - (p-1)^{N-1})}{1 - (p-1)} - N + 1 \right) \\
 &= \frac{-N(p-2) + (p-1)^N - 1}{(p-2)((p-1)^N - 1)} \\
 &= \frac{N((p-1)^N - p + 1) - N((p-1)^N - 1) + (p-1)^N - 1}{(p-2)((p-1)^N - 1)} \\
 &= \frac{N(p-1)((p-1)^{N-1} - 1) - (N-1)((p-1)^N - 1)}{(p-2)((p-1)^N - 1)}.
 \end{aligned}$$

□

From [Theorem 5.2](#), we have $H_N^{-1} = U_N^{-1} L_N^{-1} P_N^{-1}$, and so the entries of H_N^{-1} can also be calculated as follows. For $1 \leq i, j \leq N - 1$, we have

$$H_N^{-1}(i, j) = \frac{1}{(p-2)((p-1)^N - 1)} \begin{cases} (p-2)^2 & \text{if } i = j = 1, \\ (p-2)((p-1)^N - p(p-1)^{N-1} + p^2 - p - 1) & \text{if } i = j = 2, \\ -(p-2)((p-1)^{N-1} - p + 1) & \text{if } (i, j) = (1, 2), \\ p(p-2)^2 & \text{if } (i, j) = (2, 1), \\ -(p-2)((p-1)^{N-j+1} - p + 1) & \text{if } i = 1, 3 \leq j \leq N-1, \\ -p(p-2)((p-1)^{N-j+1} - p + 1) & \text{if } i = 2, 3 \leq j \leq N-1, \\ (p-2)((p-1)^i - 1) & \text{if } 3 \leq i \leq N-2, j = 1, \\ (p-2)(p(p-1)^{i-1} - (p-1)^{N-1} - 1) & \text{if } 3 \leq i \leq N-2, j = 2, \\ -((p-1)^i - 1)((p-1)^{N-j+1} - p + 1) & \text{if } 3 \leq i < j \leq N-1, \\ (p-2)((p-1)^i - 1) + ((p-1)^{-j+1} - 1)((p-1)^N - (p-1)^i) & \text{if } 3 \leq j \leq i \leq N-2, \\ (p-2)((p-1)^{N-j} - 1) & \text{if } i = N-1. \end{cases}$$

6. Application: Effective resistance and Kirchhoff index

The *effective resistance* between a pair of vertices u and v of a weighted graph $G = (V, E)$ is the electrical resistance seen between the vertices u and v of a resistor network with branch conductances given by the edge weights, denoted by $r(G; u, v)$.

In 1996, Chandra et al. [1] proved the following formula relating the effective resistances and the AHTs.

Theorem 6.1 (Chandra et al., 1996 [1]).

$$r(G; u, v) = \frac{h(G; u, v) + h(G; v, u)}{\sum_{(x, y) \in V \times V} w((x, y))}.$$

Let $V = \{v_1, v_2, \dots, v_n\}$. The *Kirchhoff index* $\text{Kf}(G)$ of the graph G is defined as $\text{Kf}(G) := \frac{1}{2} \sum_{u, v \in V} r(G; u, v)$.

We can obtain the following theorems.

Theorem 6.2. We define a weight $w(\{v_i, v_j\})$ on the weighted undirected Cayley graph $\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\})$ as follows:

$$w(\{v_i, v_j\}) = \begin{cases} p & \text{if } j - i = 1, i \text{ is even,} \\ q & \text{if } j - i = 1, i \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases}$$

where $i, j \in \mathbb{Z}$ and $v_i = i + 2n\mathbb{Z} \in \mathbb{Z}_{2n}$.

Then,

$$\text{Kf}(\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\})) = \frac{n(n-1)(n+1) + 3np(1-p)}{3p(1-p)}.$$

Proof.

$$\begin{aligned} & \text{Kf}(\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\})) \\ &= \sum_{0 \leq i < j \leq 2n-1} r(\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\}); v_i, v_j) \\ &= \sum_{0 \leq i < j \leq 2n-1} \frac{h(\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\}); i, j) + h(\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\}); j, i)}{\sum_{0 \leq k, \ell \leq 2n-1} w((v_k, v_\ell))} \\ &= \frac{1}{2n} \sum_{0 \leq i \leq 2n-1} \sum_{0 \leq j \leq 2n-1} h(\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\}); i, j) \\ &= \frac{1}{2n} \left(n \sum_{\ell=1}^{2n-1} h(\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\}); 0, \ell) + n \sum_{\ell=1}^{2n-1} h(\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\}); 1, \ell+1) \right) \\ &= \frac{1}{2} \left(\sum_{\ell=1}^{n-1} h(\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\}); 0, 2\ell) + \sum_{\ell=0}^{n-1} h(\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\}); 0, 2\ell+1) \right. \\ &\quad \left. + \sum_{\ell=1}^n h(\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\}); 1, 2\ell) + \sum_{\ell=1}^{n-1} h(\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\}); 1, 2\ell+1) \right) \\ &= \frac{1}{2p(1-p)} \left(\sum_{\ell=1}^{n-1} 2\ell(2n-2\ell) + \sum_{\ell=0}^{n-1} (2\ell(2n-(2\ell+1)+1) + 4(1-p)(n-(2\ell+1)+p)) \right. \\ &\quad \left. + \sum_{\ell=1}^n ((2\ell-2)(2n-(2\ell-1)+1) + 4p(n-(2\ell-1)+1-p)) + \sum_{\ell=1}^{n-1} 2\ell(2n-2\ell) \right) \\ &= \frac{1}{8p(1-p)} \left(8 \sum_{\ell=1}^{n-1} \ell(n-\ell) + \sum_{\ell=0}^{n-1} (4\ell(n-\ell-2(1-p)) + 4(1-p)(n-1+p)) \right. \\ &\quad \left. + \sum_{\ell=1}^n (4\ell(n-\ell+2(1-p)) - 4(n+1) + 4p(n+2-p)) \right) \\ &= \frac{1}{2p(1-p)} \left(4 \sum_{\ell=1}^{n-1} \ell(n-\ell) - \sum_{\ell=1}^n (2p^2 - 4p + 2) + 2n(1-p) \right) \\ &= \frac{1}{p(1-p)} \left(n^2(n-1) - \frac{n(n-1)(2n-1)}{3} - n(p^2 - 2p + 1) + n(1-p) \right) \\ &= \frac{n(n-1)(n+1) + 3np(1-p)}{3p(1-p)}. \end{aligned}$$

□

Remark 6.3. For an unweighted graph, the transition probabilities to the adjacent vertices are equal, which corresponds to setting $p = \frac{1}{2}$ in our weighted model. However, we must account for the difference in the scaling of the graph weights. In an unweighted graph, the total sum of the weights is $4n$, whereas our weighted model is normalized such that the total sum of the transition probabilities (weights) is $2n$. By adjusting our formula evaluated at $p = \frac{1}{2}$ with a factor of $\frac{1}{2}$ to correct this scaling difference, it coincides exactly with the unweighted Kirchhoff index of the cycle graph C_{2n} .

Theorem 6.4. We define a weight $w((v_i, v_j))$ on the weighted Cayley graph $\text{Cay}(\mathbb{Z}_N, \{+1, +2\})$ as follows:

$$w((v_i, v_j)) = \begin{cases} p & \text{if } j - i = 1, \\ q & \text{if } j - i = 2, \\ 0 & \text{otherwise,} \end{cases}$$

where $i, j \in \mathbb{Z}$ and $v_i = i + N\mathbb{Z} \in \mathbb{Z}_N$.

Then,

$$\begin{aligned} \text{Kf}(\text{Cay}(\mathbb{Z}_N, \{+1, +2\})) &= \\ &= \frac{N [(p-1)^N (3p-4-N(p-2)) - (N-1)(p-2)(2p-3) - 3p+4]}{2(p-2)^2((p-1)^N - 1)}. \end{aligned}$$

Proof.

$$\begin{aligned} &\text{Kf}(\text{Cay}(\mathbb{Z}_N, \{+1, +2\})) \\ &= \sum_{0 \leq i < j \leq N-1} r(\text{Cay}(\mathbb{Z}_N, \{+1, +2\}); v_i, v_j) \\ &= \sum_{0 \leq i < j \leq N-1} \frac{h(\text{Cay}(\mathbb{Z}_N, \{+1, +2\}); i, j) + h(\text{Cay}(\mathbb{Z}_N, \{+1, +2\}); j, i)}{\sum_{0 \leq k, \ell \leq N-1} w((v_k, v_\ell))} \\ &= \sum_{0 \leq i < j \leq N-1} \frac{h(\text{Cay}(\mathbb{Z}_N, \{+1, +2\}); i, j) + h(\text{Cay}(\mathbb{Z}_N, \{+1, +2\}); j, i)}{N} \\ &= \frac{1}{N} \left(\sum_{\ell=1}^{N-1} (N-\ell) h(\text{Cay}(\mathbb{Z}_N, \{+1, +2\}); 0, \ell) + \sum_{\ell=1}^{N-1} \ell h(\text{Cay}(\mathbb{Z}_N, \{+1, +2\}); 0, \ell) \right) \\ &= \sum_{\ell=1}^{N-1} h(\text{Cay}(\mathbb{Z}_N, \{+1, +2\}); 0, \ell) \\ &= \sum_{\ell=1}^{N-1} \frac{N(p-1)((p-1)^\ell - 1) - \ell((p-1)^N - 1)}{(p-2)((p-1)^N - 1)} \\ &= \frac{1}{(p-2)((p-1)^N - 1)} \left(N(p-1) \left(\sum_{\ell=1}^{N-1} (p-1)^\ell - \sum_{\ell=1}^{N-1} 1 \right) - ((p-1)^N - 1) \sum_{\ell=1}^{N-1} \ell \right) \\ &= \frac{1}{(p-2)((p-1)^N - 1)} \left(N(p-1) \left(\frac{(p-1)(1 - (p-1)^{N-1})}{1 - (p-1)} - N + 1 \right) - ((p-1)^N - 1) \frac{N(N-1)}{2} \right) \\ &= \frac{N}{(p-2)((p-1)^N - 1)} \times \\ &\quad \frac{2(p-1)^2 - 2(N-1)(2-p)(p-1) + (N-1)(2-p) + (p-1)^N(N(p-2) - 3p+4)}{2(2-p)} \\ &= \frac{N [(p-1)^N (3p-4-N(p-2)) - (N-1)(p-2)(2p-3) - 3p+4]}{2(p-2)^2((p-1)^N - 1)}. \end{aligned}$$

□

Remark 6.5. For an unweighted graph, the transition probabilities to the adjacent vertices are equal, which corresponds to setting $p = \frac{1}{2}$ in our weighted model. We must account for the difference in the scaling of the graph weights. In an unweighted graph, the total sum of the weights is $2N$, whereas our weighted model is normalized such that the total sum of the transition probabilities (weights) is N . By adjusting our formula evaluated at $p = \frac{1}{2}$ with a factor of $\frac{1}{2}$ to correct this scaling difference, we find that it coincides exactly with the unweighted Kirchhoff index.

Using the same approach, we can also obtain the weighted Kirchhoff index [9] and the degree Kirchhoff index [3, 11]. The resulting formulas coincide exactly with those presented above.

7. Conclusion

In this paper, we derived exact formulas for the average hitting times (AHTs) of random walks on certain weighted Cayley graphs, specifically the undirected graph $\text{Cay}(\mathbb{Z}_{2n}, \{\pm 1\})$ and the directed graph $\text{Cay}(\mathbb{Z}_N, \{+1, +2\})$. By employing matrix equations and matrix decompositions, we provided concise and fully combinatorial proofs that do not require any spectral graph theoretical arguments. Notably, for the directed case, we developed a novel method combining the Laplacian matrix approach with LU decomposition. Furthermore, as an important application of these main theorems, we utilized the AHT formulas to establish exact formulas for the Kirchhoff index of these weighted Cayley graphs.

Acknowledgment

The author would like to thank Professor Tsuyoshi Miezaki for his helpful discussions and comments regarding this research.

Declarations

Author contributions

The author contributed to the manuscript and approved the final version.

Conflict of interest

The author has no relevant financial or non-financial interests to disclose.

Data availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Funding

This work was supported by JSPS KAKENHI Grant Numbers JP23KJ2020 and JP25K23339, and a Waseda Research Institute for Science and Engineering Grant-in-Aid for Young Scientists (Early Bird).

Generative AI and AI-assisted technologies

The author(s) declare that no generative AI or AI-assisted technologies were used in the preparation of this manuscript, except for tools used solely for spelling, grammar, language editing, reference formatting, or similar routine editorial corrections.

References

- [1] A. K. Chandra, P. Raghavan, W. L. Ruzzo, R. Smolensky, and P. Tiwari, [The electrical resistance of a graph captures its commute and cover times](#), Comput. Complexity 6 (1996) 312–340.
- [2] X. Chang, H. Xu, and S.-T. Yau, [Spanning trees and random walks on weighted graphs](#), Pacific J. Math. 273(1) (2015) 241–255.
- [3] H. Chen and F. Zhang, [Resistance distance and the normalized Laplacian spectrum](#), Discrete Appl. Math. 155(5) (2007) 654–661.
- [4] Y. Doi, N. Konno, T. Nakamigawa, T. Sakuma, E. Segawa, H. Shinohara, S. Tamura, Y. Tanaka, and K. Toyota, [On the average hitting times of the squares of cycles](#), Discrete Appl. Math. 313 (2022) 18–28.
- [5] R. A. Horn and C. R. Johnson, [Matrix Analysis](#), 2nd ed., Cambridge University Press, Cambridge (2012).
- [6] G. Kirchhoff, [Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird](#), Ann. der Phys. 148(12) (1847) 497–508.
- [7] S. Klee and M. T. Stamps, [Linear algebraic techniques for weighted spanning tree enumeration](#), Linear Algebra Appl. 582 (2019) 391–402.
- [8] D. J. Kleitman and B. Golden, [Counting trees in a certain class of graphs](#), Amer. Math. Monthly 82(1) (1975) 40–44.
- [9] H. Mitsuhashi, H. Morita, and I. Sato, [The weighted Kirchhoff index of a graph](#), Linear Algebra Appl. 547 (2018) 1–18.
- [10] Y. Tanaka, [On the average hitting times of \$\text{Cay}\(\mathbb{Z}_N, \{+1, +2\}\)\$](#) , Discrete Appl. Math. 343 (2024) 269–276.
- [11] Q. Zhu and Y. Wang, [The Kirchhoff index of weighted graphs](#), Int. J. Algebra 7(5–8) (2013) 267–280.