

# On Albertson spectral properties of graphs with self-loops

Research Article

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**Abstract:** The Albertson irregularity measure is defined as  $Alb(\Gamma) = \sum_{uv \in E(\Gamma)} |d(u) - d(v)|$ . In this work, the concept of Albertson energy is extended from simple graphs to graphs with self-loops. Also the expression for the Albertson eigenvalues of a graph with self-loops are given. Some bounds on the Albertson energy of graphs with self-loops and the spread of  $Alb(\Gamma_S)$  are obtained. In the last section, the Albertson energy of complete, complete bipartite, crown and thorn graphs with self-loops are computed.

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**Keywords:** Self-loops, Albertson matrix, Albertson energy

## 1. Introduction

Let  $\Gamma = (V, E)$  be a finite, simple, undirected graph. The order and size of  $\Gamma$  are given by  $|V| = n$  and  $|E| = m$ , respectively. The number of edges incident on the vertex  $v$  is the degree of a vertex  $v$  in a graph  $\Gamma$ , denoted by  $\deg(u)$  or  $d_\Gamma(u)$ . The concept, energy of a graph, was coined by I. Gutman in 1978 as the sum of the absolute values of all the eigenvalues of a graph [10], denoted by  $E(\Gamma)$ . This definition is a general formula to calculate total  $\pi$ -electron energies of conjugated hydrocarbon molecules which was calculated by Erich Huckel in Huckel molecular orbital theory. In 2022, I. Gutman et al. broadened the idea of graph energy from simple graphs to graphs with self-loops [11]. To learn more about the extended adjacency matrix and graph energy with self-loops, readers can refer to [1–5, 7, 13–15, 17? –21].

The spread of the matrix  $A$  is given by  $S(A) = \max\{|\gamma_k - \gamma_j| : i, j = 1, \dots, n\}$ , where  $\gamma_k$ 's are the eigenvalues of matrix  $A$ . Thorn graph [8] is the graph obtained from  $\Gamma$  by attaching  $p_i$  pendant vertices to the vertex  $v_i$  of  $\Gamma$ , for  $i = 1, 2, \dots, n$ . Figure [1], [2], [3], [4] represents thorn cycle, thorn complete, thorn wheel, and thorn star graphs, respectively.

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A graph is considered regular if every vertex in it has the same degree. Calculations are frequently made easier by regularity. An irregular graph is one that has at least two distinct vertex degrees, making it non-regular. This has led to the definition of multiple irregularity measures. One such measure was proposed by Albertson in 1997 [6], which is given by  $Alb(\Gamma) = \sum_{uv(\Gamma)} |d(u) - d(v)|$ . In [12] authors introduced a new measure known as sigma index, given by  $\sigma(\Gamma) = \sum_{uv(\Gamma)} (d(u) - d(v))^2$ .

The Albertson matrix [6] of a graph  $\Gamma$  is a square matrix  $A = [a_{ij}]$  of order  $n$  given by

$$a_{ij} = \begin{cases} |d(u_i) - d(u_j)|, & \text{if } u_i \sim u_j \\ 0, & \text{if } u_i \not\sim u_j \\ 0, & \text{if } u_i = u_j. \end{cases}$$

Let  $S \subseteq V(\Gamma)$  and  $|S| = \alpha$ . Let  $\Gamma_S$  be the graph obtained from the simple graph  $\Gamma$ , by attaching a self-loop to each of its vertices belonging to  $S$ . Let  $E(\Gamma_S)$  and  $d_{\Gamma_S}(u)$  represent edge set of  $\Gamma_S$  and degree of vertex  $u$  in  $\Gamma_S$ , respectively.

In this paper, the Albertson matrix for a graph with self-loops is defined as  $Alb(\Gamma_S) = [a_{ij}]$ , where

$$a_{ij} = \begin{cases} |d(u_i) - d(u_j)|, & \text{if } u_i \sim u_j \\ 0, & \text{if } u_i \not\sim u_j \\ 1, & \text{if } u_i = u_j \text{ and } u_i \in S. \end{cases}$$

Let  $\gamma_1(\Gamma_S) \geq \gamma_2(\Gamma_S) \geq \dots \geq \gamma_n(\Gamma_S)$  be the eigenvalues of  $Alb(\Gamma_S)$ . Then, the Albertson energy of a graph with self-loops is given by

$$Alb_E(\Gamma_S) = \sum_{k=1}^n \left| \gamma_k(\Gamma_S) - \frac{\alpha}{n} \right|.$$

Let  $s_k = \left| \gamma_k(\Gamma_S) - \frac{\alpha}{n} \right|$ ,  $k = 1, 2, \dots, n$  denote the auxiliary eigenvalues of  $Alb(\Gamma_S)$ .

**Lemma 1.1.** [16] Let  $a_k$  and  $b_k$ ,  $1 \leq k \leq n$  are non-negative real numbers. Then

$$\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2 - \left( \sum_{k=1}^n a_k b_k \right)^2 \leq \frac{n^2}{4} (M_1 M_2 - m_1 m_2)^2.$$

$$M_1 = \max_{1 \leq k \leq n} a_k, M_2 = \max_{1 \leq k \leq n} b_k, m_1 = \min_{1 \leq k \leq n} a_k, m_2 = \min_{1 \leq k \leq n} b_k.$$

**Lemma 1.2.** [16] Suppose  $a_k$  and  $b_k$ ,  $1 \leq k \leq n$  are positive real numbers, then

$$\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2 \leq \frac{1}{4} \left( \sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2 \left( \sum_{k=1}^n a_k b_k \right)^2.$$

$$M_1 = \max_{1 \leq k \leq n} a_k, M_2 = \max_{1 \leq k \leq n} b_k, m_1 = \min_{1 \leq k \leq n} a_k, m_2 = \min_{1 \leq k \leq n} b_k.$$

**Lemma 1.3.** [16] Let  $a_k$  and  $b_k$ ,  $1 \leq k \leq n$  are positive real numbers, then

$$\left| n \sum_{k=1}^n a_k b_k - \sum_{k=1}^n a_k \sum_{k=1}^n b_k \right| \leq \beta(n)(A - a)(B - b),$$

where  $a, b, A$  and  $B$  are real constants i.e., for each  $k$ ,  $1 \leq k \leq n$ ,  $a \leq a_k \leq A$  and  $b \leq b_k \leq B$ . Further  $\beta(n) = n \left\lfloor \frac{n}{2} \right\rfloor \left( 1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \right)$ .

**Lemma 1.4.** [16] Let  $y = (y_k)$  and  $b = (b_k)$ ,  $k = 1, 2, \dots, n$  be real number sequence such that  $\sum_{k=1}^n |y_k| = 1$  and  $\sum_{k=1}^n y_k = 0$ . Then

$$\left| \sum_{k=1}^n b_k y_k \right| \leq \frac{1}{2} (\max_{1 \leq i \leq n} (b_k) - \min_{1 \leq i \leq n} (b_k)).$$

**Lemma 1.5.** [9] Let  $A$  be an  $n \times n$  Hermitian matrix ( $n \geq 3$ ) with eigenvalues  $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_n$ . Define  $M(A) = \{2\|A\|^2 - \frac{2}{n}(\text{tr}(A))^2\}^{\frac{1}{2}}$ . Then  $\sqrt{\frac{2}{n}}M(A) \leq S(A) \leq M(A)$ , where  $\|A\|$  is the Euclidean norm and  $\text{tr}(A)$  denote the trace of  $A$ .

**Lemma 1.6.** [22] Suppose  $p, q$  are non-negative integers, and suppose  $A, B, C, D$  are respectively  $p \times p$ ,  $p \times q$ ,  $q \times p$ , and  $q \times q$  matrices of complex numbers. Let  $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$  be a matrix of order  $(p+q)$ . If  $A$  is invertible, then  $\det(M) = \det(A)\det(D - CA^{-1}B)$ .

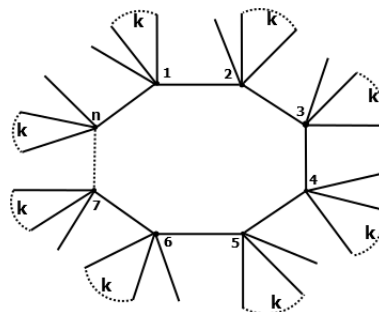


Figure 1. Thorn cycle

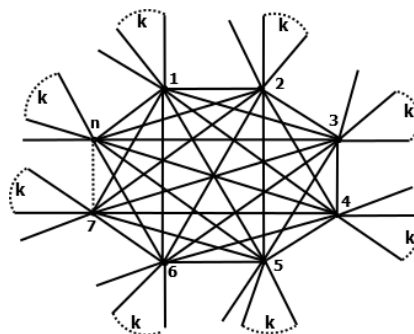


Figure 2. Thorn complete

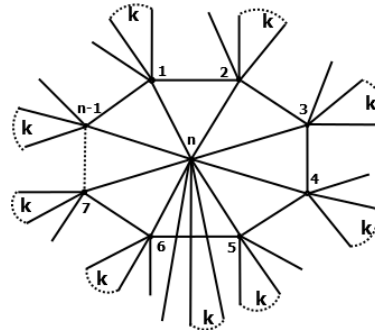


Figure 3. Thorn wheel

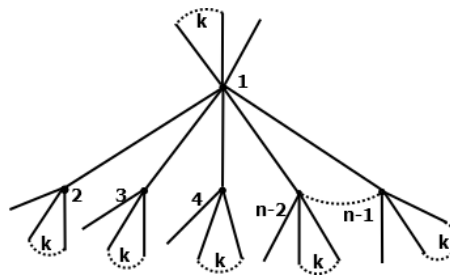


Figure 4. Thorn star

## 2. Properties

**Theorem 2.1.** Let  $\gamma_k$ ,  $1 \leq k \leq n$  be the eigenvalues of  $Alb(\Gamma_S)$ . Then

1.  $\sum_{k=1}^n \gamma_k = \alpha$ .
2.  $\sum_{k=1}^n \gamma_k^2 = 2\sigma(\Gamma_S) + \alpha$ .

**Proof.** 1. Sum of eigenvalues of  $Alb(\Gamma_S)$  is equal to trace of  $Alb(\Gamma_S)$ ,  $\sum_{k=1}^n \gamma_k = \sum_{k=1}^n a_{kk} = |S| = \alpha$ .

2. The sum of squares of eigenvalues of  $Alb(\Gamma_S)$  is the trace of  $[Alb(\Gamma_S)]^2$ .

$$\begin{aligned}
 \sum_{k=1}^n \gamma_k^2 &= \sum_{k=1}^n \sum_{j=1}^n a_{kj} a_{jk} \\
 &= \sum_{k=1}^n a_{kk}^2 + \sum_{k \neq j} a_{kj} a_{jk} \\
 &= |S| + 2\sigma(\Gamma_S) \\
 \sum_{k=1}^n \gamma_k^2 &= \alpha + 2\sigma(\Gamma_S).
 \end{aligned}$$

□

**Theorem 2.2.** Let  $s_k$ ,  $1 \leq k \leq n$  be the auxiliary eigenvalues of  $\text{Alb}(\Gamma_S)$ . Then

1.  $\sum_{k=1}^n s_k = 0$ .
2.  $\sum_{k=1}^n s_k^2 = \alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n}$ .

**Proof.** 1. We have,

$$\begin{aligned} \sum_{k=1}^n s_k &= \sum_{k=1}^n \left( \gamma_k(\Gamma_S) - \frac{|S|}{n} \right) \\ &= \sum_{k=1}^n \gamma_k(\Gamma_S) - \sum_{k=1}^n \frac{\alpha}{n} \\ &= 0. \end{aligned}$$

2. Also,

$$\begin{aligned} \sum_{k=1}^n s_k^2 &= \sum_{k=1}^n \left( \gamma_k(\Gamma_S) - \frac{|S|}{n} \right)^2 \\ &= \sum_{k=1}^n \gamma_k(\Gamma_S)^2 + \sum_{k=1}^n \left( \frac{\alpha}{n} \right)^2 - 2 \sum_{k=1}^n \gamma_k(\Gamma_S) \left( \frac{\alpha}{n} \right) \\ &= \alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n}. \end{aligned}$$

□

**Theorem 2.3.** Let  $\Gamma = (V, E)$  be a graph and  $S \subseteq V(\Gamma)$ . Let  $\Gamma_S$  be a graph with self-loops attached to all vertices of  $S$ . Then

1. If  $\alpha = 0$ ,  $\text{Alb}_E(\Gamma_S) = \text{Alb}_E(\Gamma)$ .
2. If  $\alpha = n$ ,  $\text{Alb}_E(\Gamma_S) = \text{Alb}_E(\Gamma)$ .

**Proof.** 1. If  $\alpha = 0$ , then  $\Gamma \simeq \Gamma_S$ . Therefore,  $\text{Alb}_E(\Gamma_S) = \text{Alb}_E(\Gamma)$ .

2. If  $\alpha = n$ , then  $\gamma_i(\Gamma_S) = \gamma_i(\Gamma) + 1$ .

$$\begin{aligned} \text{Alb}_E(\Gamma_S) &= \sum_{i=1}^n |\gamma_i(\Gamma_S) - 1| \\ &= \sum_{i=1}^n |\gamma_i(\Gamma) + 1 - 1| \\ &= \sum_{i=1}^n |\gamma_i(\Gamma)| \\ &= \text{Alb}_E(\Gamma). \end{aligned}$$

□

### 3. Bounds

**Theorem 3.1.** Let  $s_1 \geq s_2 \geq \cdots \geq s_n$  be the auxiliary eigenvalues of  $\text{Alb}(\Gamma_S)$ . Then

$$\text{Alb}_E(\Gamma_S) \geq \sqrt{n \left( \alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n} \right) - \frac{n^2}{4}(s_1 - s_n)^2}.$$

Equality holds if  $\Gamma \cong (K_n)_S$  with  $\alpha = n$ .

**Proof.** On substituting  $a_k = 1$  and  $b_k = |s_k|$  in Lemma [1.1], we have

$$\begin{aligned} \sum_{k=1}^n 1 \sum_{k=1}^n |s_k|^2 - \left( \sum_{k=1}^n |s_k| \right)^2 &\leq \frac{n^2}{4}(s_1 - s_n)^2 \\ n \left( \alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n} \right) - (\text{Alb}_E(\Gamma_S))^2 &\leq \frac{n^2}{4}(s_1 - s_n)^2 \\ \text{Alb}_E(\Gamma_S) &\geq \sqrt{n \left( \alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n} \right) - \frac{n^2}{4}(s_1 - s_n)^2}. \end{aligned}$$

□

**Theorem 3.2.** Let  $s_1 \geq s_2 \geq \cdots \geq s_n$  be the auxiliary eigenvalues of  $\text{Alb}(\Gamma_S)$ . Then

$$\text{Alb}_E(\Gamma_S) \geq \frac{2\sqrt{ns_1s_n \left( \alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n} \right)}}{s_1 + s_n}.$$

**Proof.** On replacing  $a_k = |s_k|$  and  $b_k = 1$  in Lemma [1.2], we obtain

$$\begin{aligned} \sum_{k=1}^n |s_k|^2 \sum_{k=1}^n 1 &\leq \frac{1}{4} \left( \sqrt{\frac{s_1}{s_n}} + \sqrt{\frac{s_n}{s_1}} \right)^2 \left( \sum_{k=1}^n |s_k| \right)^2 \\ n \left( \alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n} \right) &\leq \frac{1}{4} \frac{(s_1 + s_n)^2}{s_1 s_n} (\text{Alb}_E(\Gamma_S))^2 \\ \frac{4(s_1 s_n) n \left( \alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n} \right)}{(s_1 + s_n)^2} &\leq (\text{Alb}_E(\Gamma_S))^2 \\ \text{Alb}_E(\Gamma_S) &\geq \frac{2\sqrt{ns_1s_n \left( \alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n} \right)}}{s_1 + s_n}. \end{aligned}$$

□

**Theorem 3.3.** Let  $s_1 \geq s_2 \geq \cdots \geq s_n$  be the auxiliary eigenvalues of  $\text{Alb}(\Gamma_S)$ . Then

$$\text{Alb}_E(\Gamma_S) \geq \sqrt{n \left( \alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n} \right) - \beta(n)(|s_1| - |s_n|)^2}.$$

Equality holds if  $\Gamma \cong (K_n)_S$  with  $\alpha = n$ .

**Proof.** On substituting  $a_k = |s_k| = b_k$ ,  $a = |s_n| = b$  and  $A = |s_1| = B$  in Lemma [1.3], we have

$$\begin{aligned} \left| n \sum_{k=1}^n |s_k|^2 - \left( \sum_{k=1}^n |s_k| \right)^2 \right| &\leq \beta(n)(|s_1| - |s_n|)^2 \\ \left| n \left( \alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n} \right) - (Alb_E(\Gamma_S))^2 \right| &\leq \beta(n)(|s_1| - |s_n|)^2 \\ Alb_E(\Gamma_S) &\geq \sqrt{n \left( \alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n} \right) - \beta(n)(|s_1| - |s_n|)^2}. \end{aligned}$$

□

**Theorem 3.4.** Let  $S(Alb(\Gamma_S))$  denote the spread of  $Alb(\Gamma_S)$ . Then

$$\sqrt{\frac{2}{n}} \left( 2 \sum_{k=1}^n s_k^2 \right)^{\frac{1}{2}} \leq S(Alb(\Gamma_S)) \leq \left( 2 \sum_{k=1}^n s_k^2 \right)^{\frac{1}{2}}.$$

Left inequality is sharp if  $\Gamma$  is a  $r$ -regular graph on  $n$  vertices with consecutive  $\frac{n}{2}$  self-loops.

**Proof.** We have  $\|A\|^2 = \sigma(\Gamma_S) + \alpha$  and  $[tr(Alb(\Gamma_S))]^2 = \alpha^2$ .

On substituting this in Lemma [1.5], we have

$$\begin{aligned} \sqrt{\frac{2}{n}} \left( 2(\sigma(\Gamma_S) + \alpha) - \frac{2}{n}\alpha^2 \right)^{\frac{1}{2}} &\leq S(Alb(\Gamma_S)) \leq \left( 2(\sigma(\Gamma_S) + \alpha) - \frac{2}{n}\alpha^2 \right)^{\frac{1}{2}} \\ \sqrt{\frac{2}{n}} \left( 2 \left( \sigma(\Gamma_S) + \alpha - \frac{\alpha^2}{n} \right) \right)^{\frac{1}{2}} &\leq S(Alb(\Gamma_S)) \leq \left( 2 \left( \sigma(\Gamma_S) + \alpha - \frac{\alpha^2}{n} \right) \right)^{\frac{1}{2}} \\ \sqrt{\frac{2}{n}} \left( 2 \sum_{k=1}^n s_k^2 \right)^{\frac{1}{2}} &\leq S(Alb(\Gamma_S)) \leq \left( 2 \sum_{k=1}^n s_k^2 \right)^{\frac{1}{2}}. \end{aligned}$$

□

**Theorem 3.5.** Let  $S(Alb(\Gamma_S))$  denote the spread of  $Alb(\Gamma_S)$ . Then

$$Alb(\Gamma_S) \geq \frac{2\alpha + 4\sigma(\Gamma_S) - \frac{4\alpha^2}{n}}{S(Alb(\Gamma_S))}.$$

**Proof.** On setting  $b_k = \gamma_k$  and  $y_k = \frac{\gamma_k - \frac{\alpha}{n}}{\sum_{k=1}^n \left| \gamma_k - \frac{\alpha}{n} \right|}$ , for all  $k = 1, 2, \dots, n$  in Lemma [1.4], we obtain

$$\left| \sum_{k=1}^n b_k y_k \right| = \left| \sum_{k=1}^n \frac{\gamma_k^2 - \frac{\alpha}{n}\gamma_k}{\sum_{k=1}^n \left| \gamma_k - \frac{\alpha}{n} \right|} \right| \leq \frac{1}{2} \left( \max_{1 \leq k \leq n} \gamma_k - \min_{1 \leq k \leq n} \gamma_k \right) = \frac{1}{2} S(Alb(\Gamma_S)).$$

$$\left| \sum_{k=1}^n \frac{\gamma_k^2 - \frac{\alpha}{n} \gamma_k}{\sum_{k=1}^n |\gamma_k - \frac{\alpha}{n}|} \right| \geq \frac{\sum_{k=1}^n \gamma_k^2}{\sum_{k=1}^n |\gamma_k - \frac{\alpha}{n}|} - \frac{\left| \frac{\alpha}{n} \sum_{k=1}^n \gamma_k \right|}{\sum_{k=1}^n |\gamma_k - \frac{\alpha}{n}|} = \frac{\alpha + 2\sigma(\Gamma_S) - \frac{\alpha^2}{n} - \frac{\alpha^2}{n}}{Alb_E(\Gamma_S)}.$$

We have

$$\frac{1}{2}S(Alb(\Gamma_S)) \geq \frac{\alpha + 2\sigma(\Gamma_S) - 2\frac{\alpha^2}{n}}{Alb_E(\Gamma_S)}.$$

$$\implies Alb_E(\Gamma_S) \geq \frac{2\alpha + 4\sigma(\Gamma_S) - \frac{4\alpha^2}{n}}{S(Alb(\Gamma_S))}.$$

□

## 4. Albertson energy of some graphs with self-loops

**Theorem 4.1.** For complete graph  $K_n$  with  $\alpha \geq 1$  self-loops,

$$Alb_E(K_n)_S = \frac{2(\alpha n - \alpha^2) - n}{n} + \sqrt{1 + 16\alpha(n - \alpha)}.$$

**Proof.** For complete graph  $K_n$  with  $\alpha \geq 1$  self-loops, we have

$$Alb(K_n)_S = \begin{bmatrix} I_\alpha & 2J_{\alpha \times (n-\alpha)} \\ 2J_{(n-\alpha) \times \alpha} & 0_{(n-\alpha)} \end{bmatrix}_n,$$

where  $J$  is all 1's matrix.

Let  $W = \begin{bmatrix} Y \\ Z \end{bmatrix}$  be an eigenvector of order  $n$ , such that vector  $Y$  be of order  $\alpha$  and vector  $Z$  be of order  $n - \alpha$ . Let  $\gamma(\Gamma_S)$  be a eigenvalue of  $Alb(K_n)_S$ . Then,

$$[Alb(K_n)_S - \gamma I] \begin{bmatrix} Y \\ Z \end{bmatrix} = \begin{bmatrix} (1 - \gamma I)Y + (2J)Z \\ (2J)Y - \gamma IZ \end{bmatrix}_n. \quad (1)$$

Case 1. Let  $Y = Y_j = e_1 - e_j$ ,  $2 \leq j \leq \alpha$  and  $Z = 0_{n-\alpha \times 1}$ . Using Equation [1],  $[1 - \gamma I]Y_j + 0 = (1 - \gamma)Y_j$  then,  $\gamma = 1$  is the eigenvalue with multiplicity of at least  $\alpha - 1$  since there are  $\alpha - 1$  independent vectors of the form  $Y_j$ .

Case 2. Let  $Y = 0_k$  and  $Z = Z_j$ ,  $j = 2, 3, \dots, n - \alpha$ . Using Equation [1],  $0 - \gamma IZ_j = -\gamma Z_j$  then,  $\gamma = 0$  is the eigenvalue with multiplicity of at least  $n - \alpha - 1$  since there are  $n - \alpha - 1$  independent vectors of the form  $Z_j$ .

Case 3. Let  $Y = \frac{2(n - \alpha)}{\gamma - 1}I_\alpha$  and  $Z = I_{n-\alpha}$ . Here,  $\gamma$  denotes root of the equation,  $\gamma^2 - \gamma - 4(n - \alpha)\alpha = 0$ .



From Equation [1],

$$\begin{aligned}(2J)Y - \gamma IZ &= 2J_{(n-\alpha) \times \alpha} \frac{2(n-\alpha)}{\gamma-1} 1_\alpha - \gamma I_{n-\alpha} 1_{n-\alpha} \\ &= \left\{ \frac{4(n-\alpha)\alpha}{\gamma-1} - \gamma \right\} 1_{n-\alpha} \\ &= \left\{ \frac{\gamma^2 - \gamma - 4(n-\alpha)\alpha}{\gamma-1} \right\} 1_{n-\alpha}.\end{aligned}$$

So,  $\gamma_1 = \frac{1 + \sqrt{1 + 16(n-\alpha)\alpha}}{2}$  and  $\gamma_2 = \frac{1 - \sqrt{1 + 16(n-\alpha)\alpha}}{2}$  are the eigenvalues both with multiplicity of at least one.

The spectrum of  $Alb(K_n)_S$  is given by,

$$\begin{pmatrix} 1 & 0 & \gamma_1 & \gamma_2 \\ \alpha-1 & n-\alpha-1 & 1 & 1 \end{pmatrix}$$

where,  $\gamma_1 = \frac{1 + \sqrt{1 + 16(n-\alpha)\alpha}}{2}$ ,  $\gamma_2 = \frac{1 - \sqrt{1 + 16(n-\alpha)\alpha}}{2}$ .

The Albertson characteristic polynomial of  $Alb(K_n)_S$  is given by,

$$\gamma^{n-\alpha-1}(\gamma-1)^{\alpha-1}\{\gamma^2 - \gamma - 4(n-\alpha)\alpha\}.$$

$$\begin{aligned}Alb_E(K_n)_S &= (\alpha-1) \left| 1 - \frac{\alpha}{n} \right| + (n-\alpha-1) \left| 0 - \frac{\alpha}{n} \right| \\ &\quad + \left| \frac{1 + \sqrt{1 + 16(n-\alpha)\alpha}}{2} - \frac{\alpha}{n} \right| + \left| \frac{1 - \sqrt{1 + 16(n-\alpha)\alpha}}{2} - \frac{\alpha}{n} \right| \\ &= (\alpha-1) \left( \frac{n-\alpha}{n} \right) + (n-\alpha-1) \left( \frac{\alpha}{n} \right) + \sqrt{1 + 16\alpha(n-\alpha)} \\ &= \frac{2(\alpha n - \alpha^2) - n}{n} + \sqrt{1 + 16\alpha(n-\alpha)}.\end{aligned}$$

□

**Theorem 4.2.** For complete bipartite graph  $K_{p,q}$ ,

$$Alb_E(K_{p,q})_S = \frac{2pq - p - q}{p+q} + \sqrt{1 + 4pq(q-p+2)^2}.$$

**Proof.** For complete bipartite graph  $K_{p,q}$  with  $\alpha = p$  self-loops, we have

$$[Alb(K_{p,q})_S - \gamma I] = \begin{bmatrix} [1-\gamma]I_p & (q-p+2)J_{p \times q} \\ (q-p+2)J_{q \times p} & -\gamma I_q \end{bmatrix}_{p+q},$$

where  $J$  is matrix of all 1's.

Since block  $A = [1-\gamma]I_p$  is invertible, by Lemma [1.6], we have

$$|Alb(K_{p,q})_S - \gamma I| = |[1-\gamma]I_p| - \gamma I_q - (q-p+2)J_{q \times p} \frac{1}{1-\gamma} I_p (q-p+2)J_{p \times q}.$$

On simplifying, we obtain the characteristic polynomial of  $Alb(K_{p,q})_S$  is given by

$$\gamma^{q-1}(\gamma-1)^{p-1}\{\gamma^2 - \gamma - pq(q-p+2)^2\}.$$

The spectrum of  $Alb(K_{p,q})_S$  is given by

$$\begin{pmatrix} 0 & 1 & \gamma_1 & \gamma_2 \\ q-1 & p-1 & 1 & 1 \end{pmatrix}$$

where,  $\gamma_1 = \frac{1 + \sqrt{1 + 4pq(q-p+2)^2}}{2}$  and  $\gamma_2 = \frac{1 - \sqrt{1 + 4pq(q-p+2)^2}}{2}$ .

$$\begin{aligned} Alb_E(K_{p,q})_S &= (q-1) \left| 0 - \frac{\alpha}{p+q} \right| + (p-1) \left| 1 - \frac{\alpha}{p+q} \right| \\ &+ \left| \frac{1 + \sqrt{1 + 4pq(q-p+2)^2}}{2} - \frac{\alpha}{p+q} \right| \\ &+ \left| \frac{1 - \sqrt{1 + 4pq(q-p+2)^2}}{2} - \frac{\alpha}{p+q} \right| \\ &= (q-1) \left( \frac{p}{p+q} \right) + (p-1) \left( \frac{p}{p+q} \right) \\ &+ \sqrt{1 + 4pq(q-p+2)^2} \\ &= \frac{2pq - p - q}{p+q} + \sqrt{1 + 4pq(q-p+2)^2}. \end{aligned}$$

□

**Corollary 4.3.** For star graph,  $Alb_E(K_{1,q-1})_S = \frac{q-1}{q+1} + \sqrt{1 + 4q(q+1)^2}$ .

**Proof.** On substituting  $p = 1$  in Theorem [4.2], we obtain the required result. □

**Theorem 4.4.** For crown graph,

$$Alb_E(S_n^0)_S = (n-1)(4.0616) + \sqrt{1 + 16(n-1)^2}.$$

**Proof.** Let  $S_n^0$  be crown graph of order  $2n$  and let  $S = \{1, 2, \dots, n\}$ . Then,

$$Alb(S_n^0)_S = \begin{bmatrix} I_n & 2(J-I)_n \\ 2(J-I)_n & 0_n \end{bmatrix}_{2n},$$

is the Albertson matrix of  $(S_n^0)_S$ . The result is proved by showing

$Alb[(S_n^0)_S]Z = \gamma Z$  for certain vector  $Z$  and by making use of fact that the geometric multiplicity and algebraic multiplicity of each eigenvalue  $\gamma$  is same, as  $Alb(S_n^0)_S$  is real and symmetric.

Let  $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$  be an eigenvector of order  $2n$  partitioned conformally with  $Alb(S_n^0)_S$ .

Consider,

$$[Alb(S_n^0)_S - \gamma I] \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} (1-\gamma)IX + 2(J-I)Y \\ 2(J-I)X - \gamma IY \end{bmatrix}_{2n}. \quad (2)$$

Case 1. Let  $X = \frac{-2Y_j}{\gamma-1}$ , and  $Y = Y_j = e_1 - e_j$ ,  $j = 3, \dots, 2n$ .

Using Equation [2], we have  $\frac{1+\sqrt{17}}{2}$  and  $\frac{1-\sqrt{17}}{2}$  are the eigenvalues both with multiplicity of at least  $2n-2$  since there are  $2n-2$  independent vectors of the form  $Y_j$ .

Case 2. Let  $X = \frac{2(n-1)}{1-\gamma}1_n$  and  $Y = 1_n$ , where  $\gamma$  is any root of the Equation,

$$\gamma^2 - \gamma - 4(n-1)^2 = 0.$$

Using Equation [2],

$$\begin{aligned} 2(J-I)X - \gamma IY &= 2(J-I)\frac{2(n-1)}{1-\gamma}1_n - \gamma I1_n \\ &= \left\{ \frac{4(n-1)(1-n)}{\gamma-1} - \gamma \right\} 1_n \\ &= \left\{ \frac{4(n-1)^2}{1-\gamma} - \gamma \right\} 1_n \\ &= \left\{ \frac{\gamma^2 - \gamma - 4(n-1)^2}{\gamma-1} \right\} 1_n. \end{aligned}$$

So,  $\gamma = \frac{1+\sqrt{1+16(n-1)^2}}{2}$  and  $\gamma = \frac{1-\sqrt{1+16(n-1)^2}}{2}$  are the eigenvalues both with multiplicity of at least 1.

Thus, the spectrum of  $Alb(S_n^0)_S$  is given by,

$$\begin{pmatrix} \frac{1+\sqrt{17}}{2} & \frac{1-\sqrt{17}}{2} & \gamma_1 & \gamma_2 \\ n-1 & n-1 & 1 & 1 \end{pmatrix},$$

where  $\gamma_1 = \frac{1+\sqrt{1+16(n-1)^2}}{2}$ ,  $\gamma_2 = \frac{1-\sqrt{1+16(n-1)^2}}{2}$ .

$$\begin{aligned} Alb_E(S_n^0)_S &= (n-1) \left| \frac{1+\sqrt{17}}{2} - \frac{1}{2} \right| + (n-1) \left| \frac{1-\sqrt{17}}{2} - \frac{1}{2} \right| \\ &+ \left| \frac{1+\sqrt{1+16(n-1)^2}}{2} - \frac{1}{2} \right| + \left| \frac{1-\sqrt{1+16(n-1)^2}}{2} - \frac{1}{2} \right| \\ &= (n-1)(4.0616) + \sqrt{1+16(n-1)^2}. \end{aligned}$$

□

**Theorem 4.5.** Let  $\Gamma$  be thorn graph of order  $n(\alpha+1)$ . Let  $\Gamma_S$  be the graph obtained by attaching  $\alpha$  self-loops to all the pendant vertices of  $\Gamma$ . Then, whenever  $\Gamma$  is regular,  $Alb_E(\Gamma_S) = \frac{(\alpha-1)(n\alpha+n-\alpha)}{\alpha+1} + \sqrt{1+4\alpha N}$ , where  $N = d_\Gamma(v) + \alpha - 3$ .

**Proof.** We have,

$$[Alb(\Gamma_S) - \gamma I] = \begin{bmatrix} [(Alb(\Gamma) - \gamma I)_n]_1 & [(NI)_n]_{1 \times \alpha} \\ [(NI)_n]_{\alpha \times 1} & [(1 - \gamma I)_n]_\alpha \end{bmatrix}_{\alpha+1}, \quad (3)$$

where  $N = d_\Gamma(v) + \alpha - 3$ .

On applying row operation  $R_i \longrightarrow R_i - R_{i+1}$ ,  $2 \leq i \leq \alpha$  and column operations  $C_i \longrightarrow C_i + C_{i-1} + \dots + C_2$ ,  $2 \leq i \leq k$ , in Equation [3], we get

$$|Alb(\Gamma_S) - \gamma I| = (\gamma - 1)^{n(\alpha-1)} \begin{vmatrix} [Alb(\Gamma) - \gamma I]_n & [\alpha N I]_n \\ [N I]_n & [1 - \gamma I]_n \end{vmatrix} \quad (4)$$

Whenever  $\Gamma$  is regular, the Albertson characteristic polynomial of thorn graph with self-loops is given by  $(\gamma - 1)^{n(\alpha-1)}\{\gamma^2 - \gamma - \alpha N^2\}^n$ .

Thus, the spectrum of  $Alb(\Gamma_S)$  is given by,

$$\begin{pmatrix} 1 & \gamma_1 & \gamma_2 \\ n(\alpha - 1) & n & n \end{pmatrix},$$

where  $\gamma_1 = \frac{1 + \sqrt{1 + 4\alpha N}}{2}$ ,  $\gamma_2 = \frac{1 - \sqrt{1 + 4\alpha N}}{2}$ .

$$\begin{aligned} Alb_E(\Gamma_S) &= n(\alpha - 1) \left| 1 - \frac{\alpha}{n(\alpha + 1)} \right| + \left| \frac{1 + \sqrt{1 + 4\alpha N}}{2} - \frac{\alpha}{n(\alpha + 1)} \right| \\ &\quad + \left| \frac{1 - \sqrt{1 + 4\alpha N}}{2} - \frac{\alpha}{n(\alpha + 1)} \right| \\ &= \frac{(\alpha - 1)(n\alpha + n - \alpha)}{\alpha + 1} + \sqrt{1 + 4\alpha N}. \end{aligned}$$

□

**Corollary 4.6.** For thorn complete graph,

$$Alb_E(K_{n(\alpha+1)})_S = \frac{(\alpha - 1)(n\alpha + n - \alpha)}{\alpha + 1} + n\sqrt{1 + 4\alpha(n + \alpha - 4)^2}.$$

**Proof.** On substituting  $Alb(K_n)_S = 0$  and  $N = n + \alpha - 4$  in Equation [4], we have the Albertson characteristic polynomial of  $(K_{n(\alpha+1)})_S$  as  $(\gamma - 1)^{n(\alpha-1)}\{\gamma^2 - \gamma - \alpha(n + \alpha - 4)^2\}^n$ .

$$\begin{aligned} Alb_E(K_{n(\alpha+1)})_S &= n(\alpha - 1) \left| 1 - \frac{\alpha}{n(\alpha + 1)} \right| \\ &\quad + n \left| \frac{1 + \sqrt{1 + 4\alpha(n + \alpha - 4)^2}}{2} - \frac{\alpha}{n(\alpha + 1)} \right| \\ &\quad + \left| \frac{1 - \sqrt{1 + 4\alpha(n + \alpha - 4)^2}}{2} - \frac{\alpha}{n(\alpha + 1)} \right| \\ &= \frac{(\alpha - 1)(n\alpha + n - \alpha)}{\alpha + 1} + n\sqrt{1 + 4\alpha(n + \alpha - 4)^2}. \end{aligned}$$

□

**Corollary 4.7.** For thorn cycle,

$$Alb_E(C_{n(\alpha+1)})_S = \frac{(\alpha - 1)(n\alpha + n - \alpha)}{\alpha + 1} + n\sqrt{1 + 4\alpha(\alpha - 1)^2}.$$

**Proof.** On substituting  $Alb(C_n)_S = 0$  and  $N = \alpha - 1$  in Equation [4], we have the Albertson characteristic polynomial of  $(C_{n(\alpha+1)})_S$  is given by  $(\gamma - 1)^{n(\alpha-1)}\{\gamma^2 - \gamma - \alpha(\alpha - 1)^2\}^n$ .

$$\begin{aligned}
 Alb_E(C_{n(\alpha+1)})_S &= n(\alpha-1) \left| 1 - \frac{\alpha}{n(\alpha+1)} \right| \\
 &\quad + n \left| \frac{1 + \sqrt{1 + 4\alpha(\alpha-1)^2}}{2} - \frac{\alpha}{n(\alpha+1)} \right| \\
 &\quad + \left| \frac{1 - \sqrt{1 + 4\alpha(\alpha-1)^2}}{2} - \frac{\alpha}{n(\alpha+1)} \right| \\
 &= \frac{(\alpha-1)(n\alpha + n - \alpha)}{\alpha+1} + n\sqrt{1 + 4\alpha(\alpha-1)^2}.
 \end{aligned}$$

□

**Theorem 4.8.** The Albertson characteristic polynomial of thorn wheel graph  $W_{n(\alpha+1)}$  with  $\alpha$  self-loops is given by  $(\gamma-1)^{n(\alpha-1)}(\gamma-\gamma^2+\alpha(n+\alpha-5)^2)^{n-2}\{\gamma^4-2\gamma^3+\gamma^2[1-\alpha(a^2+b^2)-n+1]+\gamma[\alpha(a^2+b^2)-2(n+1)]+\alpha^2a^2b^2-n+1\}$ , where  $a=n+\alpha-5$  and  $b=n+\alpha-4$ .

**Proof.** From Theorem [4.5], we have

$$|Alb(W_{n(\alpha+1)})_S - \gamma I| = (\gamma-1)^{n(\alpha-1)} \begin{vmatrix} [Alb(\Gamma) - \gamma I]_n & [mNI]_n \\ [NI]_n & [1 - \gamma I]_n \end{vmatrix}.$$

On expanding the above equation,

$$\begin{aligned}
 |Alb(W_{n(\alpha+1)})_S - \gamma I| &= (\gamma-1)^{n(\alpha-1)} \{Alb(W_n)_S(1-\gamma)I - (\gamma(1-\gamma))I - \alpha N^2 I\} \\
 &= (\gamma-1)^{n(\alpha-1)} \left[ \begin{matrix} 0_{n-1} & (1-\gamma)_{n-1 \times 1} \\ (1-\gamma)_{1 \times n-1} & 0_1 \end{matrix} \right]_n - [\gamma(1-\gamma)I]_n \\
 &\quad - \left[ \begin{matrix} (\alpha(n+\alpha-5)^2)I_{n-1} & 0_{n-1 \times 1} \\ 0_{1 \times n-1} & (\alpha(n+\alpha-4)^2)I_1 \end{matrix} \right]_n \\
 [Alb(W_{n(\alpha+1)})_S - \gamma I] &= \left[ \begin{matrix} [\gamma^2 - \gamma - \alpha a^2]I_{n-1} & 0_{n-1 \times 1} \\ 0_{1 \times n-1} & [\gamma^2 - \gamma - \alpha b^2]_1 \end{matrix} \right]_n \quad (5)
 \end{aligned}$$

On applying row operation  $R_i \rightarrow R_i - R_{i+1}$ ,  $2 \leq i \leq n-1$  and column operations  $C_i \rightarrow C_i + C_{i-1} + \dots + C_1$ ,  $1 \leq i \leq n$ , in Equation [5], we get

$$[Alb(W_{n(\alpha+1)})_S - \gamma I] = (\gamma - \gamma^2 + \alpha(n + \alpha - 5)^2)^{n-2} \{(\gamma^2 - \gamma - \alpha(n + \alpha - 4)^2)(\gamma^2 - \gamma - \alpha(n + \alpha - 5)^2) - (1 - \gamma)^2(n - 1)\}.$$

The Albertson characteristic polynomial of thorn wheel graph with self-loops is given by  $(\gamma-1)^{n(\alpha-1)}(\gamma-\gamma^2+\alpha(n+\alpha-5)^2)^{n-2}\{\gamma^4-2\gamma^3+\gamma^2[1-\alpha(a^2+b^2)-n+1]+\gamma[\alpha(a^2+b^2)-2(n+1)]+\alpha^2a^2b^2-n+1\}$ . □

**Theorem 4.9.** The Albertson characteristic polynomial of thorn star graph  $K_{1,q-1}$  with  $\alpha$  self-loops is given by  $(\gamma-1)^{q(\alpha-1)}(\gamma-\gamma^2+\alpha(q+\alpha-5)^2)^{q-2}\{\gamma^4-2\gamma^3+\gamma^2[1-\alpha(a^2+b^2)-q^3+5q^2-8q+4]+\gamma[\alpha(a+b)+2q^3-10q^2+16q-8]+\alpha a^2b^2-q^3+5q^2-8q+4\}$ , where  $a=q+\alpha-5$  and  $b=q+\alpha-4$ .

**Proof.** From Theorem [4.5], we have

$$|Alb(K_{\alpha,\alpha(q-1)})_S - \gamma I| = (\gamma-1)^{q(\alpha-1)} \begin{vmatrix} (Alb(K_{1,q-1}) - \gamma I)_q & \alpha NI_q \\ NI_q & (1 - \gamma I)_q \end{vmatrix}.$$

On expanding the above equation,

$$\begin{aligned}
 [Alb(K_{\alpha, \alpha(q-1)})_S - \gamma I] &= (\gamma - 1)^{q(\alpha-1)} \{ Alb(K_{1, q-1})(1 - \gamma)I - (\gamma(1 - \gamma))I - \alpha N^2 I \} \\
 &= \begin{bmatrix} 0_1 & [(q-2)(1-\gamma)]_{1 \times q-1} \\ [(q-2)(1-\gamma)]_{q-1 \times 1} & 0_{q-1} \end{bmatrix}_q \\
 &\quad - [\gamma(1 - \gamma)I]_q - \begin{bmatrix} [\alpha(q + \alpha - 4)^2]_{q-1} & 0_{q-1 \times 1} \\ 0_{1 \times q-1} & [\alpha(q + \alpha - 5)^2]_1 \end{bmatrix}_q \\
 [Alb(K_{\alpha, \alpha(q-1)})_S - \gamma I] &= \begin{bmatrix} [\gamma^2 - \gamma - \alpha b^2]_{q-1} & 0_{q-1 \times 1} \\ 0_{1 \times q-1} & [\gamma^2 - \gamma - \alpha a^2]_1 \end{bmatrix}_q. \tag{6}
 \end{aligned}$$

On applying row operation  $R_i \rightarrow R_i - R_{i+1}$ ,  $2 \leq i \leq q-1$  and column operations  $C_i \rightarrow C_i + C_{i-1} + \dots + C_2$ ,  $2 \leq i \leq q$ , in Equation [6] we get

$$[Alb(K_{\alpha, \alpha(q-1)})_S - \gamma I] = (\gamma - \gamma^2 + \alpha(q + \alpha - 5)^2)^{q-2} \{ (\gamma^2 - \gamma - \alpha(q + \alpha - 4)^2)(\gamma^2 - \gamma - \alpha(q + \alpha - 5)^2) - (1 - \gamma)^2(q-1) \}.$$

The Albertson characteristic polynomial of thorn star with self-loops is given by

$$(\gamma - 1)^{q(\alpha-1)} (\gamma - \gamma^2 + \alpha(q + \alpha - 5)^2)^{q-2} \{ \gamma^4 - 2\gamma^3 + \gamma^2 [1 - \alpha(a^2 + b^2) - q^3 + 5q^2 - 8q + 4] + \gamma[\alpha(a + b) + 2q^3 - 10q^2 + 16q - 8] + \alpha a^2 b^2 - q^3 + 5q^2 - 8q + 4 \}. \quad \square$$

## 5. Conclusion

The concept of Albertson energy of a graph with self-loops has been defined. The set  $S \subseteq V$  that we choose affects the Albertson energy. Some bounds on  $Alb_E(G_S)$  and  $S(Alb(G_S))$  have been obtained. Additionally, the Albertson energy of a few standard graphs with self-loops is calculated. Also, considered the thorn graph with self-loops attached to the pendant vertices and compute the Albertson energy of the thorn complete, thorn cycle with self-loops. Furthermore, an expression for the Albertson characteristic polynomial for the thorn wheel and thorn star with self-loops has been provided.

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