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A supercharacter theory for $PSL(2, q)$ and $SO(3, q)$

Research Article

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Abstract: The concept of a supercharacter theory for a finite group was introduced in 2008 by Diaconis and Iasaacs in [6]. In their article the notion of irreducible characters and conjugacy classes is generalized to superchacters and superclasses while still maintaining important information about the group. This article continues an investigation of a specific supercharacter theory where the supercharacters are taken to be sums of irreducible characters of the same degree. We show this supercharacter theory construction can be done for all projective special linear groups $PSL(2, q)$ and all special orthogonal groups $SO(3, q)$ where q is any power of an (even or odd) prime.

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1. Introduction

Let q denote a power of a prime and let $\mathbf{F}_q$ denote the field of order q. Let $UT(n, q)$ denote the set of all n by n upper-triangular unipotent matrices with entries in $\mathbf{F}_q$. The notion of what later evolved into supercharacter theory was first introduced by C. André in [2], [3], [4] and by N. Yan in [14] as an approach to better understand the representation theory of $UT(n, q)$. P. Diaconis and I.M Isaacs in [6] then generalized this concept to all finite groups. A supercharacter theory of a group can be used to handle many interesting problems in character theory, without needing to know the full character table of the group.

Let G denote a finite group. Let $\hat{G}$ denote the irreducible characters of G. Let χ_0 denote the trivial character of G and let 1_G denote the identity element in G. Let $\kappa(G)$ denote a partition of the conjugacy classes of G that does not contain the empty set. Let $Ch(G)$ denote a partition of the irreducible characters of G that does not contain the empty set. The pair $(Ch(G), \kappa(G))$ is called a *supercharacter theory* for G if the following conditions hold.

1. $|Ch(G)| = |\kappa(G)|$.

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2. $\{\chi_0\} \in Ch(G)$ and $\{1_G\} \in \kappa(G)$.

3. Given any $X \in Ch(G)$, the character $\sigma_X = \sum_{\chi\in X} \chi(1)\chi$ is constant on each set in the partition $\kappa(G)$.

Given a supercharacter theory the characters $\sigma_X = \sum_{\chi\in X} \chi(1)\chi$ will be called the supercharacters and the sets in $\kappa(G)$ will be called the superclasses. We can construct the supercharacter theory table with rows the σ_X such that $X \in Ch(G)$, columns the sets in $\kappa(G)$ and entries the values of the σ_X on each set in $\kappa(G)$.

One approach in investigating supercharacter theories is to fix a family of groups and determine all the supercharacter theories this family contains. For example M.L. Lewis and C. Wynn in [10] describe the supercharacter theories of semiextraspecial p-groups and of Frobenius groups. S. Andrew in [5] describes the supercharacters theories of groups that are semidirect products with abelian normal subgroups. As another example, Hendrickson in [8] provides methods to classify all supercharacter theories of cyclic groups.

Instead of fixing the group and then determining which supercharacter theories it possesses, we alternatively could determine an algorithm for constructing the partition of irreducible characters and then determine for which groups there is a partitioning of the conjugacy classes where this creates a supercharacter theory. By Theorem 2.2 in [6] the partitions $Ch(G)$ and $\kappa(G)$, in a supercharacter theory $(Ch(G), \kappa(G))$, uniquely determine each other. Thus there will be at most one supercharacter theory that can be constructed when using a fixed algorithm for constructing the partition of irreducible characters of a group.

Given a fixed finite group G, let D denote the set of degrees of irreducible characters of G. For $d \in D$, χ_d will denote a nontrivial irreducible character of G of degree d. If G has more than one nontrivial irreducible character of degree d, we will order these characters and use the notation $\chi_{d,i}$ to denote the ith nontrivial irreducible character of degree d. The sum of all nontrivial irreducible characters of G of degree d will be denoted Ψ_d. (If there is only one character of degree d then $\Psi_d = \chi_d$.) We will continue to denote the trivial character of G by χ_0. Given $d \in D$, let X_d denote the set of all nontrivial characters of G of degree d. Let $Ch(G) = \{\{\chi_0\}, X_d \mid d \in D\}$. If a supercharacter theory exist for a given group G using this partition $Ch(G)$, we will call it the *superdegree theory* for G. Note that for this partitioning of the irreducible characters of G we have, for each $X \in Ch(G)$, that $\chi(1)$ is constant in the sum $\sigma_X = \sum_{\chi\in X} \chi(1)\chi$. Thus each σ_X is a multiple of Ψ_d for some $d \in D$. Thus we will slightly abuse terminology and call Ψ_d a superdegree character for G, even when this construction does not necessarily produce a supercharacter theory.

Note all abelian groups have a superdegree theory with supercharacter table as in Table 1. Here ρ_G denotes the regular character of G. Note also, when trying to construct a superdegree theory, it is enough to check condition 1 of the definition of as supercharacter theory holds if we force one of the superclasses to be $\{1_G\}$.

Table 1. Superdegree table for abelian groups

	$\{1\}$	$\{G$ - $\{1\}\}$
1_G	1	1
ρ_G	$\lvert G\rvert - 1$	-1

Superdegree theory was first investigated in [11] where it is shown groups that have at most seven conjugacy classes have a superdegree theory. Also in [11] it is shown that semidirect products $C_m \rtimes C_n$, where m or n is prime have a superdegree theory. The article [12] describes which Frobenius groups have a superdegree theory.

It is straightforward to check (for example in [7]) that most families of groups do not have a superdegree theory. After some background material in the following section, we demonstrate in Section 3 two families of groups that do have a superdegree theory. Specifically Section 3 is a proof of the following proposition.

Proposition 1.1. *The following families of groups have a superdegree theory for q a power of any prime.*
1) The projective special linear groups $PSL(2, q)$
2) The special orthogonal groups $SO(3, q)$.

Section 3 also presents the superdegree character tables of $PSL(2, q)$ and $SO(3, q)$. We will need to split q into the cases where $q \equiv 1 \bmod 4$, $q \equiv 3 \bmod 4$ and q even in order to construct the superdegree character tables of $PSL(2, q)$ and $SO(3, q)$. Section 4 then demonstrates that when we increase the dimension, $PSL(3, q)$ and $SO(5, q)$ do not in general have a superdegree theory.

2. Background and notation

Let $\mathbf{E} = \{x \in \mathbf{F}_{q^2} \mid x^{q+1} = 1\}$. Let η be the unique nontrivial irreducible character of $\hat{\mathbf{F}}_q^*$ such that $\eta^2 = 1$ and let η' be the unique nontrivial irreducible character of $\hat{\mathbf{E}}$ such that $(\eta')^2 = 1$. Let η_n for $1 \le n \le \frac{q-1}{2}$ denote the irreducible characters in $\hat{\mathbf{F}}^*_{\frac{q-1}{2}}$ given by $\eta_n(x) = x^n$ for x a $\frac{q-1}{2}$ root of unity. Let η'_m for $1 \le m \le (q+1)$ denote the irreducible characters in $\hat{\mathbf{E}}$ given by $\eta'_m(y) = y^m$ for y a $q+1$ root of unity. Choose Δ so that $\delta = \sqrt{\Delta}$ is a primitive $q+1$ root of unity where $\mathbf{F}_{q^2} = \mathbf{F}_q(\delta)$.

In exhibiting the superdegree character theory of several families of finite groups, we will make frequent use of the following two lemmas.

Lemma 2.1. *Let ζ be a primitive 2ℓ root of unity for some positive integer ℓ and fix a positive integer j. Then*

$$\sum_{m=1}^{\ell-1} (\zeta^{jm} + \zeta^{-jm}) = \begin{cases} 0 & \text{if } j \text{ is odd} \\ -2 & \text{if } j \text{ is even} \end{cases}.$$

Proof. Note that $\zeta^{-jm} = (\zeta^{-m})^j = (\zeta^{2\ell-m})^j$. As m ranges from 1 to $\ell - 1$ we have that $2\ell - m$ ranges from $2\ell - 1, 2\ell - 2, \ldots, 2\ell - (\ell - 1)$. That is, $2\ell - m$ ranges from $\ell + 1$ up to $2\ell - 1$. Thus

$$\sum_{m=1}^{\ell-1} (\zeta^{jm} + \zeta^{-jm}) = \sum_{m=1}^{\ell-1} \zeta^{jm} + \sum_{m=\ell+1}^{2\ell-1} \zeta^{jm} = (\sum_{m=1}^{2\ell} \zeta^{jm}) - \zeta^{2j\ell} - \zeta^{j\ell}.$$

Since ζ is a primitive 2ℓ root of unity, $\sum_{m=1}^{2\ell} \zeta^{jm} = 0$ and $\zeta^{2\ell} = 1$ and $\zeta^{\ell} = -1$. Thus

$$\sum_{m=1}^{\ell-1} (\zeta^{jm} + \zeta^{-jm}) = 0 - 1 - (-1)^j = \begin{cases} 0 & \text{if } j \text{ is odd} \\ -2 & \text{if } j \text{ is even} \end{cases}.$$

□

Lemma 2.2. *Let ζ be a primitive $2\ell + 1$ root of unity for some positive integer ℓ. Then for a fix positive integer j we have*

$$\sum_{m=1}^{\ell} (\zeta^{jm} + \zeta^{-jm}) = -1.$$

Proof. Note that $\zeta^{-jm} = (\zeta^{-m})^j = (\zeta^{2\ell+1-m})^j$. As m ranges from 1 to ℓ we have that $2\ell+1-m$ ranges from $2\ell, 2\ell-1, \dots, 2\ell+1-\ell$. That is, $2\ell+1-m$ ranges from $\ell+1$ up to 2ℓ. Thus

$$\sum_{m=1}^{\ell}(\zeta^{jm}+\zeta^{-jm}) = \sum_{m=1}^{\ell}\zeta^{jm} + \sum_{m=\ell+1}^{2\ell}\zeta^{jm} = (\sum_{m=1}^{2\ell+1}\zeta^{jm}) - \zeta^{j(2\ell+1)}.$$

Since ζ is a primitive $2\ell+1$ root of unity, $\sum_{m=1}^{2\ell+1}\zeta^{jm} = 0$ and $\zeta^{2\ell+1} = 1$. Thus $\sum_{m=1}^{\ell}(\zeta^{jm}+\zeta^{-jm}) = -1$. □

3. Families of groups that have a superdegree theory

3.1. PSL(2,q) for q congruent to 1 mod 4

The generic character tables for $PSL(2,q)$ for q odd is presented in [9]. A version of this table with more modern notation can be found at [1]. Using a mixture of the notation presented in these two citations, there is an ordering of the conjugacy classes of $PSL(2,q)$ for $q \equiv 1 \bmod 4$, so that its generic character table is the $\frac{q+5}{2}$ by $\frac{q+5}{2}$ Table 2. Here t is a primitive $\frac{q+1}{2}$ root of unity, r is a primitive $\frac{q-1}{2}$

Table 2. Character table of $PSL(2,q)$ for $q \equiv 1 \bmod 4$

	I	A_1	A_2	B_j	C_k	D
χ_0	1	1	1	1	1	1
$\chi_{q-1,m}$	$q-1$	-1	-1	$-t^{mj}-t^{-mj}$	0	0
χ_q	q	0	0	-1	1	1
$\chi_{q+1,n}$	$q+1$	1	1	0	$r^{nk}+r^{-nk}$	$2\eta_n(\sqrt{-1})$
$\chi_{(q+1)/2,1}$	$\frac{q+1}{2}$	$\frac{1+\sqrt{q}}{2}$	$\frac{1-\sqrt{q}}{2}$	0	$\eta(r^k)$	$\eta(\sqrt{-1})$
$\chi_{(q+1)/2,2}$	$\frac{q+1}{2}$	$\frac{1-\sqrt{q}}{2}$	$\frac{1+\sqrt{q}}{2}$	0	$\eta(r^k)$	$\eta(\sqrt{-1})$

root of unity, $1 \le j, m \le \frac{q-1}{4}$ and $1 \le n, k \le \frac{q-5}{4}$. The character η of $\mathbf{F}_q^*$ and characters η_n of $\mathbf{F}^*_{\frac{q-1}{2}}$ are as described in Section 2 above. (When $q = 5$ there are no irreducible characters of degree $q+1$ and there are no conjugacy classes C_k.)

In order to construct the superdegree theory we need to add together the characters of degree $q-1$, those of degree $q+1$, and the two characters of degree $\frac{q+1}{2}$.

Using that $q-1 \equiv 0 \bmod 4$, we have $q-1 = 4\ell$ for some positive integer ℓ. Thus $q+1 = 4\ell+2$ and thus $\frac{q+1}{2} = 2\ell+1$. Note that then $\frac{q-1}{4} = \frac{4\ell+1-1}{4} = \ell$. Thus we have by Lemma 2.2 that

$$\sum_{m=1}^{\frac{q-1}{4}}(-t^{mj}-t^{-mj}) = 1.$$

Thus when adding the characters of the degree $q-1$ into a single supercharacter Ψ_{q-1} and taking the union of all conjugacy classes B_j to obtain the single superclass $B = \cup_j B_j$ we get Table 3.

Note since $q \equiv 1 \bmod 4$ we have that 4 divides $q-1$. That is $\frac{q-1}{2}$ is even. Let $2\ell = \frac{q-1}{2}$ in Lemma 2.1. Then $\frac{q-5}{4} = \frac{4\ell-4}{4} = \ell - 1$. Thus

$$\sum_{n=1}^{\frac{q-5}{4}}(r^{nk}+r^{-nk}) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ -2 & \text{if } k \text{ is even} \end{cases}.$$

Table 3. Building a superdegree table for $PSL(2,q)$ for $q \equiv 1 \bmod 4$, Part 1

	I	A_1	A_2	B	C_k	D
χ_0	1	1	1	1	1	1
Ψ_{q-1}	$\frac{(q-1)^2}{4}$	$\frac{-(q-1)}{4}$	$\frac{-(q-1)}{4}$	1	0	0
χ_q	q	0	0	-1	1	1
$\chi_{q+1,n}$	$q+1$	1	1	0	$r^{nk}+r^{-nk}$	$2\ \eta_n(\sqrt{-1})$
$\chi_{(q+1)/2,1}$	$\frac{q+1}{2}$	$\frac{1+\sqrt{q}}{2}$	$\frac{1-\sqrt{q}}{2}$	0	$\eta(r^k)$	$\eta(\sqrt{-1})$
$\chi_{(q+1)/2,2}$	$\frac{q+1}{2}$	$\frac{1-\sqrt{q}}{2}$	$\frac{1+\sqrt{q}}{2}$	0	$\eta(r^k)$	$\eta(\sqrt{-1})$

Furthermore as $\eta \in \hat{\mathbf{F}}_q^*$ is the unique nontrivial degree one character such that $\eta^2 = 1$, we have that $\eta(r) = -1$ and thus

$$\eta(r^k) = \begin{cases} -1 & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even} \end{cases}.$$

Let $C_{odd} = \cup C_k$ for k odd and let $C = \cup C_k$ for k even. Then adding the characters of degree $q+1$ together we can take C_{odd} and C_{even} as superclasses. In addition add the two characters of degree $\frac{q+1}{2}$ and note that we can then take $A = A_1 \cup A_2$ as a superclass, giving us Table 4.

Table 4. Building a superdegree table for $PSL(2,q)$ for $q \equiv 1 \bmod 4$, Part 2

	I	A	B	C_{odd}	C_{even}	D
χ_0	1	1	1	1	1	1
Ψ_{q-1}	$\frac{(q-1)^2}{4}$	$\frac{-(q-1)}{4}$	1	0	0	0
χ_q	q	0	-1	1	1	1
Ψ_{q+1}	$\frac{(q+1)(q-5)}{4}$	$\frac{q-5}{4}$	0	0	-2	$2\sum_n \eta_n(\sqrt{-1})$
$\Psi_{(q+1)/2}$	$q+1$	1	0	-2	2	$2\eta(\sqrt{-1})$

Note that $\eta(-1) = (-1)^{\frac{q-1}{2}}$ and thus $\eta(\sqrt{-1}) = (-1)^{\frac{q-1}{4}}$. Thus

$$\eta(\sqrt{-1}) = \begin{cases} -1 & \text{if } \frac{q-1}{4} \text{ is odd} \\ 1 & \text{if } \frac{q-1}{4} \text{ is even} \end{cases}.$$

Finally note that

$$2\sum_n \eta_n(\sqrt{-1}) = 2\sum_{n=1}^{\frac{q-5}{4}} (-1)^n = \begin{cases} 0 & \text{if } \frac{q-5}{4} \text{ is even} \\ -2 & \text{if } \frac{q-5}{4} \text{ is odd} \end{cases} = \begin{cases} 0 & \text{if } \frac{q-1}{4} \text{ is odd} \\ -2 & \text{if } \frac{q-1}{4} \text{ is even} \end{cases}.$$

Thus the column labeled with conjugacy class D is the same as the column labeled by C_{odd} when $\frac{q-1}{4}$ is odd otherwise column D is the same as column C_{even}. Let $C' = C_{odd} \cup D$ and $C'' = C_{even}$ when $\frac{q-1}{4}$ is odd. Let $C' = C_{odd}$ and $C'' = C_{even} \cup D$ when $\frac{q-1}{4}$ is even.

Thus the superdegree character table of $PSL(2,q)$ for $q \equiv 1 \bmod 4$ is the 5 by 5 Table 5.

When $q = 5$ the 4 by 4 superdegree table is Table 5 with the supercharacter Ψ_{q+1} removed, the superclass C'' removed, and the superclass $C' = D$.

Table 5. The superdegree table of $PSL(2,q)$ for $q \equiv 1 \bmod 4$

	I	A	B	C'	C''
χ_0	1	1	1	1	1
Ψ_{q-1}	$\frac{(q-1)^2}{4}$	$\frac{-(q-1)}{4}$	1	0	0
χ_q	q	0	-1	1	1
Ψ_{q+1}	$\frac{(q+1)(q-5)}{4}$	$\frac{q-5}{4}$	0	0	-2
$\Psi_{(q+1)/2}$	q+1	1	0	-2	2

3.2. PSL(2,q) for q congruent to 3 mod 4

We again refer to the generic character tables for $PSL(2,q)$ for q odd presented in [9] and [1]. Again using a merger of the presentation of the generic character tables from these citations, there is an ordering of the conjugacy classes of $PSL(2,q)$ for $q \equiv 3 \bmod 4$, so that its generic character table is Table 6. Here

Table 6. Character table of $PSL(2,q)$ for $q \equiv 3 \bmod 4$

	I	A_1	A_2	B_j	C_k	D
χ_0	1	1	1	1	1	1
$\chi_{q-1,m}$	$q-1$	-1	-1	$-t^{mj} - t^{-mj}$	0	$-2\eta'_m(\delta)$
χ_q	q	0	0	-1	1	-1
$\chi_{q+1,n}$	$q+1$	1	1	0	$r^{nk} + r^{-nk}$	0
$\chi_{(q-1)/2,1}$	$\frac{q-1}{2}$	$\frac{-1+\sqrt{-q}}{2}$	$\frac{-1-\sqrt{-q}}{2}$	$-\eta'(t^j)$	0	$-\eta'(\delta)$
$\chi_{(q-1)/2,2}$	$\frac{q-1}{2}$	$\frac{-1-\sqrt{-q}}{2}$	$\frac{-1+\sqrt{-q}}{2}$	$-\eta'(t^j)$	0	$-\eta'(\delta)$

t is a primitive $\frac{q+1}{2}$ root of unity, r is a primitive $\frac{q-1}{2}$ root of unity and $1 \le m,n,j,k \le \frac{q-3}{4}$. When $q=3$ that are no irreducible characters of degree $q-1$ or of degree $q+1$. (For $q \equiv 3 \bmod 4$ there are two typographic errors in [1]. The corrections are the value of χ_q on the conjugacy D is -1 and the number of conjugacy classes B_j is $\frac{q-3}{4}$.)

Taking $q-3 = 4(\ell-1)$ for some positive integer $\ell > 1$, we have $q+1 = 4\ell$. So $\frac{q+1}{2}$ is even. Thus we can take $2\ell = \frac{q+1}{2}$ in Lemma 2.1 so that $\frac{q-3}{4} = \ell - 1$; giving us

$$\sum_{m=1}^{\frac{q-3}{4}} (t^{jm} + t^{-jm}) = \begin{cases} 0 & \text{if } j \text{ is odd} \\ -2 & \text{if } j \text{ is even} \end{cases}.$$

Also, as $\eta' \in \hat{\mathbf{E}}$ is the unique nontrivial degree one character such that $(\eta')^2 = 1$, we have that $\eta'(t) = -1$ and thus

$$\eta'(t^j) = \begin{cases} -1 & \text{if } j \text{ is odd} \\ 1 & \text{if } j \text{ is even} \end{cases}.$$

Thus when we sum all the characters of degree $q-1$ to form a single supercharacter and also sum the two characters of degree $\frac{q-1}{2}$ together, we can create two superclasses $B_{odd} = \cup B_j$, for j odd, and $B_{even} = \cup B_j$, for j even; and we can combine the conjugacy classes A_1 and A_2 into a single superclass, A, giving us Table 7.

Table 7. Building a superdegree table for $PSL(2,q)$ for $q \equiv 3 \bmod 4$

	I	A	B_{odd}	B_{even}	C_k	D
χ_0	1	1	1	1	1	1
Ψ_{q-1}	$\frac{(q-1)(q-3)}{4}$	$\frac{-(q-3)}{4}$	0	2	0	$-2\sum_m \eta'_m(\delta)$
χ_q	q	0	-1	-1	1	-1
$\chi_{q+1,n}$	$q+1$	1	0	0	$r^{nk}+r^{-nk}$	0
$\Psi_{(q-1)/2}$	$q-1$	-1	2	-2	0	$-2\eta'(\delta)$

Since $q-3=4\ell$ for some positive integer ℓ, we have $q-1=4\ell+2$ and thus $\frac{q-1}{2}$ is odd. So we can take $2\ell+1=\frac{q-1}{2}$ in Lemma 2.2 and then $\frac{q-3}{4}=\ell$ and thus

$$\sum_{n=1}^{\frac{q-3}{4}} r^{nk}+r^{-nk}=-1.$$

Thus adding all the characters of degree $q+1$ allows us to form a single superclass $C=\cup_k C_k$.

Note that $\eta'(\Delta)=(-1)^{\frac{q+1}{2}}$ and thus $\eta'(\delta)=(-1)^{\frac{q+1}{4}}$. Thus

$$-2\eta'(\delta)=\begin{cases} 2 & \text{if } \frac{q+1}{4} \text{ is odd} \\ -2 & \text{if } \frac{q+1}{4} \text{ is even}\end{cases}.$$

Finally note that

$$-2\sum_m \eta'_m(\delta)=-2\sum_{m=1}^{\frac{q-3}{4}}(-1)^m=\begin{cases} 0 & \text{if } \frac{q-3}{4} \text{ is even} \\ 2 & \text{if } \frac{q-3}{4} \text{ is odd}\end{cases}=\begin{cases} 0 & \text{if } \frac{q+1}{4} \text{ is odd} \\ 2 & \text{if } \frac{q+1}{4} \text{ is even}\end{cases}.$$

Thus the column labeled with conjugacy class D is the same as the column labeled by B_{odd} when $\frac{q+1}{4}$ is odd otherwise column D is the same as column B_{odd}. Let $B'=B_{odd}\cup D$ and $B''=B_{even}$ when $\frac{q+1}{4}$ is odd. Let $B'=B_{odd}$ and $B''=B_{even}\cup D$ when $\frac{q+1}{4}$ is even.

Thus the superdegree character table of $PSL(2,q)$ for $q\equiv 3 \bmod 4$ is Table 8.

Table 8. The superdegree table of $PSL(2,q)$ for $q \equiv 3 \bmod 4$

	I	A	B'	B''	C
χ_0	1	1	1	1	1
Ψ_{q-1}	$\frac{(q-1)(q-3)}{4}$	$\frac{-(q-3)}{4}$	0	2	0
χ_q	q	0	-1	-1	1
Ψ_{q+1}	$\frac{(q+1)(q-3)}{4}$	$\frac{q-3}{4}$	0	0	-1
$\Psi_{(q-1)/2}$	$q-1$	-1	-2	2	0

Since, when $q=3$ that are no irreducible characters of degree $q-1$ or of degree $q+1$, the superdegree character table when $q=3$ is 3 by 3 as there is no superclass C or B'' and superclass $B'=D$.

The case were $G=PSL(2,q)$ where q is even will be considered in the next section.

3.3. SO(3,q) for q even

In this subsection we assume q is a power of 2 and construct the superdegree theory of the group $SO(3,q)$. For the remainder of this article, we will use the notation ζ_n to denote a fixed primitive nth root of unity.

The generic character table of $SO(3,q)$ for q even can be found in [13]. There is an ordering of the conjugacy classes so that this character table is Table 9 with some notational changes from [13]. Here

Table 9. Character table of $SO(3,q)$ for q even

	I	A	B_j	C_k
χ_0	1	1	1	1
χ_q	q	0	1	-1
$\chi_{q-1,n}$	$q-1$	-1	0	$-(\zeta_{q+1}^{kn}+\zeta_{q+1}^{knq})$
$\chi_{q+1,m}$	$q+1$	1	$\zeta_{q-1}^{jm}+\zeta_{q-1}^{-jm}$	0

$1 \le m, j \le \frac{1}{2}(q-2)$ and $1 \le n, k \le \frac{q}{2}$.

In order to construct the superdegree character theory for $SO(3,q)$ we will need to sum together the characters of degree $q-1$ and the characters of degree $q+1$.

Lemma 3.1. *For $1 \le j \le \frac{1}{2}(q-2)$ and $1 \le k \le \frac{q}{2}$*

$$\sum_{m=1}^{\frac{q-2}{2}} \zeta_{q-1}^{jm}+\zeta_{q-1}^{-jm} = -1 \quad \textit{and} \quad \sum_{n=1}^{\frac{q}{2}} -(\zeta_{q+1}^{kn}+\zeta_{q+1}^{knq}) = 1.$$

Proof. Since q is even both $q-1$ and $q+1$ are odd. First let $2\ell+1 = q-1$. Then $\ell = \frac{q-2}{2}$ and the first summation follows from Lemma 2.2. Next take $2\ell+1 = q+1$. Then $\ell = \frac{q}{2}$ and then second summation follows again from Lemma 2.2, noting that when ζ has order $q+1$ then $\zeta^{-1} = \zeta^q$. □

Thus we can combine all the conjugacy classes B_j into a single superclass B and we can combine all the conjugacy classes C_k into a single superclass C. Thus summing together the characters of degree $q-1$ and the characters of degree $q+1$ we get the superdegree character table for $SO(3,q)$ (q even) is Table 10. In this table, $B = \cup B_j, C = \cup C_k$. (When $q = 2$ there are no irreducible characters of degree

Table 10. The superdegree table of $SO(3,q)$ for q even

	I	A	B	C
χ_0	1	1	1	1
χ_q	q	0	1	-1
Ψ_{q-1}	$\frac{q(q-1)}{2}$	$-\frac{q}{2}$	0	1
Ψ_{q+1}	$\frac{(q+1)(q-2)}{2}$	$\frac{q-2}{2}$	-1	0

$q+1$ and the superclass B is removed.) Thus $SO(3,q)$ for q even has a superdegree theory.

When q is a power of 2, we have that $SO(3,q) \cong SL(2,q) \cong PSL(2,q) \cong PGL(2,q)$. Thus combining the results in Subsections 3.1, 3.2, and 3.3, we have $PSL(2,q)$ has a superdegree theory for any q. More generally, we have $PGL(n,q) \cong SL(n,q) \cong PSL(n,q)$ when the greatest common divisor of n and $q-1$

equals 1. Thus $SL(2, q)$ has a superdegree theory if $gcd(2, q-1) = 1$. However it is not the case that $SL(2, q)$ will always have a superdegree theory. For example consider the group $SL(2, 5)$. There is an ordering of the conjugacy classes of $SL(2, 5)$ so that its character table is Table 11 [7].

Table 11. Character table of $SL(2, 5)$

	I	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
χ_0	1	1	1	1	1	1	1	1	1
$\chi_{2,1}$	2	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	-2	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	-1	1	0
$\chi_{2,2}$	2	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	-2	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	-1	1	0
$\chi_{3,1}$	3	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	3	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0	0	-1
$\chi_{3,2}$	3	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	3	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	0	0	-1
$\chi_{4,1}$	4	-1	-1	4	-1	-1	1	1	0
$\chi_{4,2}$	4	1	1	-4	-1	-1	1	-1	0
χ_5	5	0	0	5	0	0	-1	-1	1
χ_6	6	-1	-1	-6	1	1	0	0	0

Adding the characters of like degree in an attempt to form the superdegree character table we obtain Table 12. Thus there are six superdegree characters. Note there are seven distinct columns in Table 12

Table 12. The superdegree characters of $SL(2, 5)$

	I	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
χ_0	1	1	1	1	1	1	1	1	1
Ψ_2	4	1	1	-4	-1	-1	-2	2	0
Ψ_3	6	1	1	6	1	1	0	0	-2
Ψ_4	8	0	0	0	-2	-2	2	0	0
χ_5	5	0	0	5	0	0	-1	-1	1
χ_6	6	-1	-1	-6	1	1	0	0	0

as columns C_2 and C_3 are the same and columns C_5 and C_6 are the same. Thus the minimum number of superclasses is seven. Thus $SL(2, 5)$ does not have a superdegree theory.

3.4. SO(3,q) for q odd

Now let q be a power of an odd prime number. The generic character table of $SO(3, q)$ for q odd can be found in [13]. There are two typographical errors in [13]: the number of characters of degree $q-1$ is $\frac{q-1}{2}$ not $\frac{q+1}{2}$ and the number of characters of degree $q+1$ is $\frac{q-3}{2}$ not $\frac{q-1}{2}$. There is an order of the conjugacy classes of $SO(3, q)$ so that the generic character table is Table 13. Here $1 \leq m \leq \frac{1}{2}(q-3)$ and $1 \leq n, j \leq \frac{1}{2}(q-1)$ and $1 \leq k \leq \frac{1}{2}(q+1)$.

In order to construct the superdegree character theory for $SO(3, q)$ for q odd, we will need to sum together the characters of degree $q-1$, the characters of degree $q+1$, and the two characters of degree q.

Lemma 3.2. *Fix integers j, k such that $1 \leq j \leq \frac{q-1}{2}$ and $1 \leq k \leq \frac{q+1}{2}$. Then*

Table 13. Character table of $SO(3,q)$ for q odd

	I	A	B_j	C_k
χ_0	1	1	1	1
χ_1	1	1	$(-1)^j$	$(-1)^k$
$\chi_{q,1}$	q	0	1	-1
$\chi_{q,2}$	q	0	$(-1)^j$	$(-1)^{k+1}$
$\chi_{q-1,n}$	$q-1$	-1	0	$-(\zeta_{q+1}^{kn}+\zeta_{q+1}^{knq})$
$\chi_{q+1,m}$	$q+1$	1	$\zeta_{q-1}^{jm}+\zeta_{q-1}^{-jm}$	0

$$(i) \quad \sum_{m=1}^{\frac{q-3}{2}} \zeta_{q-1}^{jm}+\zeta_{q-1}^{-jm} = \begin{cases} 0 & \text{if } j \text{ is odd} \\ -2 & \text{if } j \text{ is even} \end{cases}$$

and

$$(ii) \quad \sum_{n=1}^{\frac{q-1}{2}} -(\zeta_{q+1}^{kn}+\zeta_{q+1}^{knq}) = \begin{cases} 0 & \text{if } j \text{ is odd} \\ 2 & \text{if } j \text{ is even} \end{cases}.$$

Proof. Since $q-1$ is even, let $2\ell = q-1$. Then $\frac{q-3}{2} = \ell - 1$ and thus (i) follows from Lemma 2.1. Similarly, letting $2\ell = q+1$ we have $\frac{q-1}{2} = \ell - 1$ so (ii) also follows from Lemma 2.1. □

Thus, using also that $(-1)^\ell + 1$ is zero when ℓ is odd and is 2 when ℓ is even, we get Table 14 when characters of the same degree are added together.

Table 14. The superdegree characters of $SO(3,q)$ for q odd

	I	A	B_j j odd	B_j j even	C_k k odd	C_k k even
χ_0	1	1	1	1	1	1
χ_1	1	1	-1	1	-1	1
Ψ_q	$2q$	1	0	2	0	-2
Ψ_{q-1}	$\frac{(q-1)^2}{2}$	$-\frac{(q-1)}{2}$	0	0	0	2
Ψ_{q+1}	$\frac{(q+1)(q-3)}{2}$	$\frac{q-3}{2}$	0	-2	0	0

Combine all the conjugacy class B_j when j is even into single superclass and all the conjugacy classes C_k when k is even into a single superclass. In addition, we can combine all the conjugacy classes B_j and C_k when j,k are both odd into a single superclass, as these columns of Table 14 are equal. Thus we get the 5 by 5 superdegree character table, Table 15, for $SO(3,q)$ when q is odd. (When $q=3$ there are no irreducible characters of degree $q+1$ and the column $\cup B_j$, for j even, is removed.)

Combining the results of Subsections 3.1, 3.2, 3.3, and 3.4, we have the families $SO(3,q)$ and $PSL(2,q)$ have a superdegree theory for any q.

Table 15. The superdegree table of $SO(3,q)$ for q odd

	I	A	$B_j \cup C_k$ j,k odd	$\cup B_j$ j even	$\cup C_k$ k even
χ_0	1	1	1	1	1
χ_1	1	1	-1	1	1
Ψ_q	$2q$	0	0	2	-2
Ψ_{q-1}	$\frac{(q-1)^2}{2}$	$-\frac{(q-1)}{2}$	0	0	2
Ψ_{q+1}	$\frac{(q+1)(q-3)}{2}$	$\frac{q-3}{2}$	0	-2	0

4. Extending to higher dimensions

It is natural to wonder at this point, if we can increase the dimensions of any of the families considered in Section 3, will we still have a superdegree theory.

First consider $PSL(3,q)$ where q is odd. There is an ordering of the conjugacy classes of $PSL(3,3)$ so that its character table is Table 16 [7]. Here $A = \zeta_{13}^2 + \zeta_{13}^5 + \zeta_{13}^6, B = \zeta_{13} + \zeta_{13}^3 + \zeta_{13}^9, C = \zeta_{13}^7 + \zeta_{13}^8 + \zeta_{13}^{11}$

Table 16. Character table of $PSL(3,3)$

	I	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
χ_0	1	1	1	1	1	1	1	1	1	1	1	1
χ_{12}	12	3	0	-1	-1	-1	-1	4	1	0	0	0
χ_{13}	13	4	1	0	0	0	0	-3	0	-1	-1	1
$\chi_{16,1}$	16	-2	1	A	B	C	D	0	0	0	0	0
$\chi_{16,2}$	16	-2	1	C	D	A	B	0	0	0	0	0
$\chi_{16,3}$	16	-2	1	D	A	B	C	0	0	0	0	0
$\chi_{16,4}$	16	-2	1	B	C	D	A	0	0	0	0	0
$\chi_{26,1}$	26	-1	-1	0	0	0	0	2	-1	0	0	2
$\chi_{26,2}$	26	-1	-1	0	0	0	0	-2	1	$-\sqrt{-2}$	$\sqrt{-2}$	0
$\chi_{26,3}$	26	-1	-1	0	0	0	0	-2	1	$\sqrt{-2}$	$-\sqrt{-2}$	0
χ_{27}	27	0	0	1	1	1	1	3	0	-1	-1	-1
χ_{39}	39	3	0	0	0	0	0	-1	-1	1	1	-1

and $D = \zeta_{13}^4 + \zeta_{13}^{10} + \zeta_{13}^{12}$.

Adding the characters of like degree in an attempt to form the superdegree character table we obtain Table 17. Thus there are seven superdegree characters. There are eight distinct columns in Table 17 and thus the minimum number of superclasses is eight. Thus $PSL(3,3)$ does not have a superdegree theory.

We have $PGL(3,3) \cong SL(3,3) \cong PSL(3,3)$ since $gcd(3,3-1) = 1$. Thus the families $PSL(3,q)$, $PGL(3,q)$ and $SL(3,q)$ for q odd do not in general have a superdegree character theory. Thus the results of Subsections 3.1 and 3.2 do not extend to higher dimensions.

We also have that the results of Subsection 3.3 do not extend to $PSL(3,q)$, $PGL(3,q)$ or $SL(3,q)$ when q is even. This can be shown by looking at a portion of the character table of $PSL(3,8)$ (which is isomorphic to $PGL(3,8)$ and $SL(3,8)$ since $gcd(3,8-1) = 1$). $PSL(3,8)$ has seventy-two irreducible characters and these characters come in eight different degrees, only one (the trivial character) of which is of degree one. So $PSL(3,8)$ has eight superdegree characters. There is one irreducible character of degree 72 and six different irreducible characters of degree 73. There is an ordering of the conjugacy

Table 17. The superdegree characters of $PSL(3,3)$

	I	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
χ_0	1	1	1	1	1	1	1	1	1	1	1	1
χ_{12}	12	3	0	-1	-1	-1	-1	4	1	0	0	0
χ_{13}	13	4	1	0	0	0	0	-3	0	-1	-1	1
Ψ_{16}	64	-8	4	-1	-1	-1	-1	0	0	0	0	0
Ψ_{26}	78	-3	-3	0	0	0	0	-2	1	0	0	2
χ_{27}	27	0	0	1	1	1	1	3	0	-1	-1	-1
χ_{39}	39	3	0	0	0	0	0	-1	-1	1	1	-1

classes so that these rows of the character table are as in Table 18 [7]. Here $1 \leq r \leq 6, 1 \leq i \leq 24, 1 \leq$

Table 18. The degree 72 and 73 irreducible characters of $PSL(3,8)$

	I	A	B	C_i	D_j	E_j	F_k	G_i	H_l	J_m
χ_{72}	72	8	0	-1	9	1	0	0	2	2
$\chi_{73,r}$	73	9	1	0	$V_{j,r}$	$W_{j,r}$	1	$X_{i,r}$	$Y_{l,r}$	$Z_{m,r}$

$j \leq 6, 1 \leq k \leq 4, 1 \leq l \leq 3$ and $1 \leq m \leq 2$ and the entries $V_{j,r}, W_{j,r}, X_{i,r}, Y_{l,r}$ and $Z_{m,r}$ will be described below.

Each $V_{j,r}$ is an element in $\{\zeta_7^3 + 9\zeta_7^2, \zeta_7^2 + 9\zeta_7^6, \zeta_7^6 + 9\zeta_7^4, \zeta_7^4 + 9\zeta_7^5, \zeta_7^5 + 9\zeta_7, \zeta_7 + 9\zeta_7^3\}$ and as r ranges from 1 to 6, $V_{j,r}$ obtains each element in this set. Thus

$$\sum_r V_{j,r} = \sum_{r=1}^{6} 9\zeta_7^r + \sum_{r=1}^{6} \zeta_7^r = -10.$$

Furthermore, each $W_{j,r}$ is an element in $\{\zeta_7 + \zeta_7^3, \zeta_7 + \zeta_7^5, \zeta_7^2 + \zeta_7^3, \zeta_7^2 + \zeta_7^6, \zeta_7^4 + \zeta_7^5, \zeta_7^4 + \zeta_7^6\}$ and as r ranges from 1 to 6, $W_{j,r}$ obtains each element in this set. Thus

$$\sum_r W_{j,r} = 2\sum_{r=1}^{6} \zeta_7^r = -2.$$

The elements $X_{i,r}$ come from the set $\{\zeta_7, \zeta_7^2, \zeta_7^3, \zeta_7^4, \zeta_7^5, \zeta_7^6\}$ and as r ranges from 1 to 6, $X_{i,r}$ obtains each element in this set. Thus $\sum_r X_{i,r} = -1$.

The elements $Y_{l,r}$ come from the set $\{-\zeta_7 - \zeta_7^2 - \zeta_7^5 - \zeta_7^6,\ -\zeta_7 - \zeta_7^3 - \zeta_7^4 - \zeta_7^6,\ -\zeta_7^2 - \zeta_7^3 - \zeta_7^4 - \zeta_7^5\}$ and as r ranges from 1 to 6, $Y_{l,r}$ obtains each element in this set twice. Thus

$$\sum_r Y_{l,r} = 2\sum_{r=1}^{6} -2\zeta_7^r = 4.$$

Finally each $Z_{m,r}$ is $\frac{-1\pm\sqrt{-7}}{2}$ and as r ranges from 1 to 6, $Z_{m,r}$ obtains each of these two elements three times. Thus $\sum_r Z_{m,r} = -3$.

Thus summing all the characters of degree 73 we obtain Table 19. As nine of these ten columns are

Table 19. Two superdegree characters of $PSL(3,8)$

	I	A	B	C_i	D_j	E_j	F_k	G_i	H_l	J_m
χ_{72}	72	8	0	-1	9	1	0	0	2	2
Ψ_{73}	438	54	6	0	-10	-2	6	-1	4	-3

distinct, just looking at these two superdegree characters reveals the number of superclasses has to be greater than or equal to nine. As there are eight different degrees of characters in $PSL(3,8)$ and only one character of degree one, there are eight superdegree characters. Thus $PSL(3,8)$ does not have a superdegree theory.

The superdegree character theory of $SO(3,q)$ also does not extend to a superdegree character theory of $SO(5,q)$. For example the character table of $SO(5,3)$ has irreducible characters of twelve different degrees including two characters of degree one. So there are thirteen superdegree characters. There is an ordering of the twenty-five conjugacy class of $SO(5,3)$ so that the two irreducible characters of degree six and the only irreducible character of degree ten are as in Table 20 [7].

Table 20. The degree 6 and 10 irreducible characters of $SO(5,3)$

$\chi_{6,1}$	6	-2	2	0	-3	1	-1	0	-2	-4	2	3	-1
$\chi_{6,2}$	6	-2	2	0	-3	1	-1	0	-2	4	2	3	-1
χ_{10}	10	-6	2	0	1	-3	-1	4	0	0	2	-2	2

$\chi_{6,1}$	-1	1	0	-2	2	-1	2	1	1	0	0	0
$\chi_{6,2}$	1	1	0	2	-2	1	-2	1	-1	0	0	0
χ_{10}	0	0	1	0	0	0	0	0	0	-2	0	0

Summing the two characters of degree six we obtained Table 21.

Table 21. Two superdegree characters of $SO(5,3)$

Ψ_6	12	-4	4	0	-6	2	-2	0	-4	0	4	6	-2
χ_{10}	10	-6	2	0	1	-3	-1	4	0	0	2	-2	2

Ψ_6	0	2	0	0	0	0	0	2	0	0	0	0
χ_{10}	0	0	1	0	0	0	0	0	0	-2	0	0

As only the columns in grey are repeats of previous columns, the number of superclasses must be greater than or equal to fourteen. Thus $SO(5,3)$ does not have a superdegree theory, showing that the family $SO(5,q)$ for q a power of an odd prime does not in general have a superdegree theory.

Finally ruling at the possibility that $SO(5,q)$ has a superdegree theory when q is even, consider $SO(5,4)$. Looking at the characters of degree 18 and 34 in $SO(5,4)$ will reveal that family $SO(5,q)$ also does not in general have a superdegree theory for q even in an argument analogous to the argument where $G = SO(5,3)$ above.

Thus $PSL(3,q)$ and $SO(5,q)$ do not in general have a superdegree theory.

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References

[1] J. Adams, Character tables for $GL(2), SL(2), PGL(2)$ and $PSL(2)$ over a finite field, (2002) http://www2.math.umd.edu/ jda/characters/.

[2] C.A.M André, Basic characters of the unitriangular group, *J. Algebra* **175** (1995) 287-319.

[3] C.A.M André, The basic character table of unitriangular group, *J. Algebra* **241** (2001) 437-471.

[4] C.A.M André, Basic characters of the unitriangular group (for arbitrary primes), *Proc. Amer. Math. Soc.* **130** No. 7 (2002) 1943-1954.

[5] S. Andrews, Supercharacter theories constructed by the method of little groups, *Comm. in Algebra* **44** (2016) 2152-2179.

[6] P. Diaconis and I.M. Isaacs, Supercharacters and superclasses for algebra groups, *Trans. Amer. Math. Soc.*, **360** (2008) 2359-2392.

[7] The GAP group, *GAP - Groups, algorithms, and programming, Version 4.11.1*, (2021) https://www.gap-system.org.

[8] A.O.F. Hendrickson, *Supercharacter theories of cyclic p-groups*, Ph.D. Thesis, University of Wisconsin, (2008).

[9] H. E. Jordan, Group-characters of various types of linear groups, *Amer. J. of Math.*, **49** No. 4 (1907) 387-405.

[10] M.L. Lewis and C. Wynn, Supercharacter theories of semiextraspecial p-groups and Frobenius groups, *J. Algebra* **502** (2018) 372-388.

[11] J. Rainbolt, A Supercharacter theory based on the degrees of irreducible characters, *Algebraic Combin.*, **55** (2022) 1225-1249.

[12] J. Rainbolt, A Supercharacter theory and Frobenius groups, submitted 2023.

[13] R. Steinberg, The representations of $GL(3,q), GL(4,q), PGL(3,q)$ and $PGL(4,q)$, *Canad. J. of Math.*, **3** (1951) 225-235.

[14] N. Yan, *Representation theory of the finite unipotent linear groups*, Ph.D. Thesis, Department of Mathematics, University of Pennsylvania, (2001).