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New Results and Bounds on Codes over $GF(19)$

Research Article

Saurav R. Pandey, Nuh Aydin, Eric Z. Chen, Fredrik Jönsson, Kamilla Klonowska

Abstract: Explicit construction of linear codes over finite fields is one of the most important and challenging problems in coding theory. Due to the centrality of this problem, databases of best-known linear codes (BKLCs) over small finite fields have been available. Recently, new databases for BKLCs over larger alphabets have been introduced. In this work, a new database of BKLCs over the field $GF(19)$ is introduced, containing lower and upper bounds on the minimum distances for codes with lengths up to 150 and dimensions between 3 and 6. Computer searches were conducted on cyclic, constacyclic, quasi-cyclic, and quasi-twisted codes to establish lower bounds. These searches resulted in many new linear codes over $GF(19)$.

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1. Introduction

Let $[n,k,d]_q$ represent a linear code over the finite field $GF(q)$ with length n, dimension k, and minimum distance (weight) d. Constructing linear codes with best possible values of the parameters is a fundamental problem in coding theory [27] It is a discrete optimization problem where given the alphabet and the values of the other two parameters, we want to find the best value for the third parameter of a linear code. For instance, we might want to either minimize the block length n for a given dimension k and minimum distance d, or maximize the minimum distance d for a given block length n and dimension k. Let $d_q(n,k)$ represent the largest value of d for which there exists an $[n,k,d]_q$ code over $GF(q)$, and let $n_q(k,d)$ represent the smallest value of n for which there exists an $[n,k,d]_q$ code over $GF(q)$. An

Department of Computer Science, The University of North Carolina at Chapel Hill (email: srpandey@cs.unc.edu)
Nuh Aydin(Corresponding Author), Department of Mathematics and Statistics, Kenyon College, Gambier, Ohio, USA (email: aydinn@kenyon.edu)
Department of Computer Science, Kristianstad University, Kristianstad, Sweden (email: eric.chen@hkr.se)
Department of Computer Science, Kristianstad University, Kristianstad, Sweden (email: fredrik.jonsson@hkr.se)
Department of Computer Science, Kristianstad University, Kristianstad, Sweden (email: kamilla.klonowska@hkr.se)

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$[n, k, d]$ code is called length-optimal if its block length n is equal to $n_q(k, d)$. Similarly, it is called distance-optimal if its minimum distance d equals $d_q(n, k)$.

This optimization problem is very hard to solve. Generally, it is only solved when either k or $n-k$ are relatively small. Searching for codes with the best parameters, even with the help of modern computers, is computationally taxing. There are two main reasons for this. First, computing the minimum distance of an arbitrary linear code is NP-hard [28], and it becomes infeasible for large dimensions. Second, for a given length, dimension, and finite field $GF(q)$, the number of linear codes is very large. Due to these inherent difficulties, researchers often focus on special types of linear codes that are known to contain good codes, and at the same time possess rich mathematical structures. Cyclic codes and their various generalizations play an important role in this regard. Certain generalizations of cyclic codes such as quasi-cyclic (QC) and quasi-twisted (QT) codes are known to contain many linear codes with good parameters. In fact, many record-breaking QC and QT codes have been obtained with the help of computer search algorithms. For a sample of publications that present new linear codes from the classes of QC and QT codes, see [1–3, 9, 19, 20, 22, 23].

The online database [21] is well known in the coding theory research community for the best-known linear codes (BKLC) over small fields. MAGMA software also has a similar database included [8]. The best-known QC and QT codes are stored in the online database of QT codes [11]. As new codes are discovered, these databases are updated. The two databases in [21] and [8] store codes over finite fields having sizes up to 9. Similarly, a smaller database of codes over $GF(11)$ and $GF(13)$ are given in [12] and [13]. More recently, codes over $GF(17)$ have been studied, and a new database of codes over $GF(17)$ has been created in [14]. A construction of sector-disk codes and MDS codes over $GF(17)$ for correcting sector erasure errors is given in [15]. Computer searches on QT codes over $GF(17)$ and $GF(19)$ are conducted in [24], and codes with dimensions up to 5 are presented. Additionally, some results on self-dual codes over $GF(19)$ can be found in [7, 16], and [26]. The main goal of this paper is to introduce a new database of BKLCs over $GF(19)$ with lower and upper bounds on minimum distances.

2. Quasi-Twisted codes

Quasi-twisted (QT) codes are generalization of cyclic, constacyclic and quasi-cyclic codes. A linear code is called constacyclic if whenever a codeword $(a_0, a_1, ..., a_{n-1})$ is in the code, so is $(\alpha a_{n-1}, a_0, ..., a_{n-2})$, where α is a non-zero element in the field $GF(q)$ and is called the shift constant. The special case of $\alpha = -1$ is known as negacyclic codes which were defined in [6]. Therefore, a QT code with $p = 1$ and $\alpha = 1$ is a cyclic code; a QT code with $p = 1$ is a constacyclic code; a QT code with $\alpha = 1$ is a QC code. The block length n of a QT code can be written $n = m \cdot p$.

Given a generator polynomial $c(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$ of a constacyclic code with shift constant a, it has a generator matrix of the following a-circulant form:

$$G = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ ac_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ ac_{n-2} & ac_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ ac_{n-k+1} & ac_{n-k+2} & ac_{n-k+3} & \cdots & c_{n-k} \end{bmatrix}$$

As a generalization of constacyclic codes, a generator matrix of a QT code consists of blocks of constacyclic matrices. In general, a generator matrix of an p-QT code has the following form

$$G = \begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1p} \\ G_{21} & G_{22} & \cdots & G_{2p} \\ \vdots & \vdots & & \vdots \\ G_{r1} & G_{r2} & \cdots & G_{rp} \end{bmatrix}$$

where each G_{ij} is an a-circulant matrix corresponding to a constacyclic code. Such a code is called an r-generator QT code. The special case $[G_1\ G_2\ ...\ G_p]$ gives rise to 1-generator QT codes.

3. Computer search algorithms and best-known QT codes over $GF(19)$

We found many good QT codes using the generalized version of the ASR search algorithm [4], which is based on the notion of code equivalence.

Definition 3.1. *Two linear codes are equivalent if one code can be obtained from the other by any combination of the following transformations:*

1. *A permutation of the coordinates.*
2. *Multiplying the elements in a fixed position by a non-zero scalar in GF(q).*
3. *Applying an automorphism of GF(q) to each component of the vectors.*

The algorithm given in [4] partitions constacyclic codes of a given length into equivalence classes and then selects one code from each class. This algorithm is faster and more efficient than the command in MAGMA to test for the equivalence of linear codes. For more details on the performance and implementation of this algorithm, see [4]. After producing constacyclic codes quickly from this algorithm, we can proceed to implement the ASR algorithm to search for QT codes.

In the ASR search algorithm, the first step is to take a generator $g(x)$, $g(x) \mid x^m - a$, of a constacyclic code of length m from each equivalence class. Next, we construct a QT code of index p from a generator in the form

$$(f_1(x)g(x), f_2(x)g(x), ..., f_p(x)g(x)),$$

where all $f_i(x)$'s are randomly chosen from $\mathbb{F}_q[x]/\langle x^m - a\rangle$ with the condition that they are relatively prime to the check polynomial $h(x)$ of the constacyclic code generated by $g(x)$, and $\deg(f_i(x)) < \deg(h(x))$. The ASR algorithm is based on the following theorem.

Theorem 3.2. [2] *Let C be a 1-generator, p-QT code over $\mathbb{F}_q$ of length $n = mp$ with a generator $G(x)$ of the form:*

$$G(x) = (f_1(x)g(x), f_2(x)g(x), ..., f_p(x)g(x)),$$

where $x^m - a = g(x)h(x)$ and for all $i = 1..., p$, $\gcd(h(x), f_i(x)) = 1$. Then, C is an $[n, k, d']_q$-code where $k = m - deg(g(x))$, and $d' \geq p \cdot d$, d being the the minimum distance of the constacyclic code C_g of length m generated by $g(x)$.

Another algorithm for finding good QT codes is the iterative heuristic search algorithm, presented in [10]. It is based on the weight matrix as defined in [22]. However, for a code with large parameters k and p over large field, the weight matrix becomes too big to complete the search within a reasonable time or even fit in memory. Therefore, for these cases, we apply a randomized algorithm by choosing up to 500 defining polynomials randomly, to reduce the search space and run time complexity. For small finite fields, a larger number of random polynomials can be used, due to the smaller number of rows in the weight matrix.

With the algorithms mentioned above, a lot of good QT codes have been found. Table 1 presents the QT $[pk, k]$ codes of dimensions 3, 4, 5 and 6 for $p = 2, 3, \ldots, 25$ over $GF(19)$. The superscript e indicates the codes that improve best-known results, and the superscript o indicates optimal codes.

Table 1: Best-known QT $[pk, k]$ codes of dimensions $3, 4, 5$ and 6

p\k	3	4	5	6
2	4^o	5^o	6^o	7^o
3	7^o	9^o	11^o	13^o
4	10^o	13^o	15	18
5	13^o	17	19	22
6	16^o	20^o	24^e	28
7	18^o	24^{oe}	28	33
8	21^o	27	33^e	38
9	24^o	31^e	37	44
10	27^o	34	42^e	49
11	29	38	46	54
12	32	42^e	51^e	60
13	35	45	55	65
14	38^o	49	60^e	71
15	41^o	53^e	65^e	76
16	44^{oe}	56	69^e	81
17	47^{oe}	60	73	87
18	49	64^e	78^e	92
19	52	68^e	82	98
20	55^o	72^e	88^e	104
21	58^{oe}	75^e	92^e	109
22	61^{oe}	79^e	96	114
23	63	82	100	120
24	66	86	106^e	125
25	69	90^e	109	131

Table 2 below lists new improved QT $[pk, k]$ codes over $GF(19)$, obtained with the algorithms mentioned above. The defining polynomials are represented by the sequences of their coefficients, with the lowest coefficients on the left. The elements of the sequence are denoted by 0 to 9 and A to I, there A represents 10, B represents 11 and so on. As an example, $3AEA$ corresponds to the polynomial $3 + 10x + 14x^2 + 10x^3$.

Table 3 presents the new QT $[pk, k]$ codes with $k = 6$, and Table 4 lists the QT codes used to get the lower minimum distance bounds given in Table 5.

4. Lower and upper bounds on minimum distances of linear codes over $GF(19)$

4.1. Lower bounds on minimum distance

Lower bounds on minimum distances of linear codes are usually established by explicit constructions. Implementing the search algorithms described in the previous section, we have been able to construct many QT codes with good parameters. In the previous section, we presented tables of new codes that are explicitly constructed, so they establish the lower bounds on the minimum distances.

4.2. Upper bounds on minimum distance

Upper bounds on minimum distances are obtained by applying the standard bounds (such as Griesmer, Elias, Sphere Packing, and others [27]) and taking the best result for each parameter set. This is

Table 2: New QT $[pk, k]$ codes with improvements

[n, k, d] code	α	Defining polynomials
[48, 3, 44]	1	521, 821, 391, 6G1, H41, 8B1, 7A1, H1, 371, 941, 921, G1, A1, 321, 831, E31
[51, 3, 47]	1	521, 821, 391, 6G1, H41, 8B1, 7A1, H1, 371, 941, 921, G1, A1, 321, 831, E31
[63, 3, 58]	1	E21, 11, C11, 91, 371, IB1, H81, 541, 741, 61, E11, 351, G51, H41, H1, 911, EC1, D1, 761, B41, B21
[66, 3, 61]	1	731, 51, 821, F21, EC1, B11, F71, 31, IA1, 81, 721, G1, FA1, 941, 621, 311, 741, 411, 6E1, B41, 11, B21
[28, 4, 24]	1	I71, 2811, 5B11, 9831, 7E31, G741, 6451
[36, 4, 31]	1	221, A81, D91, BC1, 1I1, G321, BG21, C131, H481
[48, 4, 42]	1	151, IH1, 7I1, G511, 7421, DI21, I531, 6831, F451, 6E61, F781, 6E81
[60, 4, 53]	1	121, C41, C81, I111, 5811, C321, F621, CE21, DF21, HF31, G251, 3851, E861, I191, E6A1
[72, 4, 64]	1	GE1, B611, FC11, BF11, E721, H721, IC21, CI21, CE31, B541, 4B41, 9261, 2I71, 7481, 2981, E191, E591, 7E91
[76, 4, 68]	1	FD1, C711, 4121, EC21, 7141, B741, CF41, 4H41, 7851, 9I51, F861, G961, EA61, IA61, B171, 6871, 95A1, IAB1, IDB1
[80, 4, 72]	2	976B, 2581, A41, 3E9, G2F, IH7C, 47IH, AF8B, 8E71, 48C, E468, IIBC, 3AAF, F971, B4G, FBIE, 8GA9, 7598, FA11, FGI
[84, 4, 75]	1	741, A71, DB1, I611, 8C11, 8721, 4821, BD21, 5F21, 8F21, 4G21, 5341, CF41, F451, GB51, CH51, I161, 9A61, H871, E481, 68F1
[88, 4, 79]	1	41, 331, 761, 2B1, 5B1, 3111, 9411, D411, G611, 2I11, 3I11, FC21, 2431, 2C31, CI31, B241, 2H41, 6H41, E571, 3H71, E5B1, 87C1
[100, 4, 90]	2	976B, 2581, A41, 3E9, G2F, 9487, BCEC, C991, DIA, 0G5, IBEI, DH3D, 3DA1, GD, GI1, 2D93, BF8B, 7A2B, 96A1, 3FH, DG6H, GFF9, 5H6G, EIC1, 94I1
[30, 5, 24]	1	1DB, BD1, A3021, C83C, 339C2, BFF61
[40, 5, 33]	1	1DB, BD1, IIFC, 9B41, B3F71, BFDH, F3612, EIHF1
[50, 5, 42]	1	2F73, E931, E8H81, A8FF, 9G7G1, 22I8, ICH95, D1FC1, HE7H3, 8GIF1
[60, 5, 51]	1	15ID, 3AEA, AEA3, DI51, 91981, 90B8, H39I, D5HC, 34AC1, A8EB6, H4261, 289B
[70, 5, 60]	1	58EA2, HID91, F2651, 3GIA1, 2AA91, GDBA1, 8DA42, EIDA1, DBI61, G7BF, 3HD63, 51E21, 84231, E5F3
[75, 5, 65]	1	28H18, CDFE1, AH54, FIG21, 3E961, 6EGG, 2D4CD, D3DE1, 8I1H, B1GB7, 78FE2, HEHG1, BA659, GB491, 4G6E1
[80, 5, 69]	1	IH223, 464HB, 0E351, 6D5I, I9B84, 57GH5, FH3E1, BCEC1, 4601G, I47E3, 1F911, I93G, A7E4C, I339E, ICF11, 6D72
[90, 5, 78]	1	20704, D41D1, I833, A3E8H, 885B1, DECD, 35HG3, FIIB1, D192, I9EG6, 6I8B1, 155E, 85HG4, ID5C2, 2A4E1, 2DEF8, 51611,2D2G
[100, 5, 88]	1	G74E3, H5EGB, I52I1, IFEG, 2226A, G5A46, F44H1, 32II, 79807, 2AII2, CBDD1, D71A, FEC17, 674HF, 77961, D14C, C6B6I, 91122, 01851, H43A
[105, 5, 92]	1	52F4A, 79F41, 85BB1, 4AD59, DC3C1, 516A, I9EG6, 6I8B1, 155E, 85HG4, ID5C2, 2A4E1, 34FD6, 73831, 3BCE1, 2DEF8, 51611, 2D2G, 9B2AH, 98571, F267
[120, 5, 106]	1	DID17, 85FI6, 2EF11, 2651, B699G, 8H7I1, H465, AA93, I9B84, 57GH5, FH3E1, BCEC1, BF0G1, CAE07, DF1D1, 36E1, E0HGH, F44E8, 723D1, 452F, 6A9I1, 32GI5, HEID1, BI2D

similar to the case of previously introduced databases over $GF(11), GF(13)$ and $GF(19)$ ([12], [13],[14]).

4.3. Linear codes with dimension 3

It is well known that there are connections between BKLCs and projective geometry. An (n, r)-arc in $PG(k-1, q)$ is a set of n points K with the property that every hyperplane is incident with at most r points of K and there is some hyperplane incident with exactly r points of K. A linear code is called projective if any two of its coordinates are linearly independent, or equivalently if the minimum distance of its dual code is at least three. It is well known that existence of a projective $[n, 3, d]_q$ code is equivalent to the existence an $(n, n-d)$-arc in $PG(2, q)$. A Griesmer code is a code that meets the Griesmer bound (Theorem 24, [27]). Every $[n, k, d]_q$ Griesmer code with $d \leq q^{k-1}$ is projective ([25], [17], [18]). An online table of bounds on the sizes of (n, r)-arcs in $PG(2, q)$ is maintained in [5]. This allows us to obtain bounds

Table 3: New QT $[6p, 6]$ codes over $GF(19)$

[n, k, d] code	α	**Defining polynomials**
[12, 6, 7]	1	1, FB6211
[18, 6, 12]	1	BC3D31, 2B7C1, 87I711
[24, 6, 18]	2	IB2D95, 3ADFA, FH6E3B, HGA999
[30, 6, 22]	1	1, HEFG11, 3H271, B9F111, I57A91
[36, 6, 28]	1	ADCF75, F20601, I9EH4, HF973A, CHFF11, C1028
[42, 6, 33]	2	IB2D95, H80AEC, E17F1G, 9307H3, GABF61, 2FE7EA, 4CEADA
[48, 6, 38]	18	8DI74H, 96IA1A, 26DHAG, 31BD7, EF7DIC, 71H6F2, 7479C6, EE9H02
[54, 6, 44]	1	I4473C, C14EG1, DA57D, EG2745, A8C981, F707, 4FI941, GC8IF1, 0HE5F
[60, 6, 49]	8	F3C748, F0HF2F, GB834, 7253C1, B91B2, 8I3984, 6D68EE, C7DB1G, 22D621, 34BEE
[66, 6, 54]	1	I6B911, 83361, 3H271, G3G711, E2C161, F4A351, E86C91, 26B731, ID6851, 8IGH41, 387A71
[72, 6, 60]	1	7I971, 4946, BC1I, B5619G, BGFAE1, FECFB, 275G8G, DC4F91, G32C11, 4FI941, GC8IF1, 0HE5F
[78, 6, 65]	2	IB2D95, 3ADFA, FH6E3B, CGF9GI, H457A3, I0GI6, 70H4F, 6C96A5, 650BHD, 82IAG5, I9101G, HB13CD, 85GIGF
[84, 6, 71]	2	DCBHEB, CE02C4, 0HHG28, 71FEH, 33I26C, F029H, 0ADF9B, 10IBH4, DB003H, 7G02B1, 6GAD43, 521996, 59B2HE, C8BB2G
[90, 6, 76]	18	GCC61, D0AH82, D3G855, 580168, B8DA7F, F36IAD, 72DH24, 185FH2, 83228C, D78E33, 3H97GA, 5G6513, 07D03D, 00I33, 70D73B
[96, 6, 81]	18	7FA8I3, 841EGB, B6B3HA, 52D6F9, G1E73B, 17DACB, EA0597, EE6BG6, 407EGF, 5IE4B1, 81G2B, 1EGE3D, 8F7754, H5I705, 2D1BH9, DA2F6D
[102, 6, 87]	2	IB2D95, 3ADFA, FH6E3B, CGF9GI, H457A3, I0GI6, 70H4F, 6C96A5, 650BHD, 82IAG5, I9101G, HB13CD, C772GD, 1GH6DB, BGH42F, 57G31I, G4B59G
[108, 6, 92]	1	243FGD, B9H628, 0289IG, 1BDABE, 0DF8DB, 0DFEI, 9F575D, C16826, 65093C, HAFG1B, E91H46, 147DDD, G98C2B, C15678, 8GD06F, 5186A, 43FB3D, 8G402
[114, 6, 98]	2	IB2D95, DD44F8, 7HAA88, 03BH26, G5F5A2, BAA173, GIAB63, AC1HHG, 70H9B7, 6D952F, 091HG3, 57B0AC, E29143, 13G967, 8856F7, A87AD1, 18075C, ACD0GF, A2248D
[120, 6, 104]	8	979F8A, 53465C, 2I44D2, 84H6B1, 37B3I1, 121H6E, E31543, 1I5865, 941E71, 888GG, 19G6I, HFA25B, 58EF3C, BD4991, A6I55, 707D0B, F95CF3, H1F5H9, EI5F11, 3D87B
[126, 6, 109]	2	78E61B, 11GFC1, DG6C6, 3DG0D, 9A98A, I7HGA, 9D15E, G4E45C, HAAD86, 8E4E7G, DCG266, 2CE79B, 1H2B51, 6DHE5, D69EBF, 95E97H, 7BFCCH, B34E6E, 265H61, 8FB8G, H1ICG
[132, 6, 114]	2	IB2D95, 3ADFA, FH6E3B, CGF9GI, H457A3, I0GI6, 70H4F, 6C96A5, 650BHD, 82IAG5, I9101G, HB13CD, C772GD, 1GH6DB, BGH42F, 57G31I, H5IC11, CFI3A1, 386EIC, 071G97, 36AA54, 412E7
[138, 6, 120]	2	IB2D95, 3ADFA, FH6E3B, CGF9GI, H457A3, I0GI6, 70H4F, 6C96A5, 650BHD, 82IAG5, I9101G, HB13CD, C772GD, 1GH6DB, BGH42F, 57G31I, H5IC11, CFI3A1, 386EIC, 071G97, 36AA54, 7G2I72, 6AHI21
[144, 6, 125]	1	I4473C, C14EG1, DA57D, EG2745, A8C981, F707, 4FI941, GC8IF1, 0HE5F, 7I971, 4946, BC1I, I77956, AE5H51, E2BFA, C9B78D, G95CH1, 18GF1, 6DDGIE, FG69E1, IFA7A1, 308CG1, 8GF4F1, HD2FC
[150, 6, 131]	8	F3C748, F0HF2F, GB834, 7253C1, B91B2, 8I3984, 6D68EE, C7DB1G, 22D621, 34BEE, 7HC29, 8B177, 10I52, 5C928, 9BHC1, 4HB6E8, EH82C6, 211994, BA9I31, 0GEAC, 27F6E1, EHG41D, 2F4H21, 8H8CD1, I6E441

on linear codes of dimension 3 from the bounds on the sizes of (n, r)-arcs in $PG(2, q)$, and by the method of puncturing. It turns out that most of the QC or QT codes of dimension 3 presented in this work meet the bounds, hence they are optimal.

Table 5 below summarizes all of the results that we have obtained. For each length n ($n \leq 150$) and dimension k ($3 \leq k \leq 6$), it presents the lower bound and the upper bound for an $[n, k]$ code over $GF(19)$. If the minimum distance of the BKLC for that length and dimension meets the theoretical upper bound with equality, an optimal code is known and there is a single number for that entry (e.g., there is a single number $d = 18$, for $n = 20, k = 3$). If there is a gap between these two numbers then they are listed for that entry. For example, we have $18 - 19$, for $n = 22, k = 4$. This means that for a linear code of length

Table 4: New QT codes used to get the lower minimum distance bounds

n	**k**	**d**	**m**	**α**	**Defining polynomials**
50	4	44	10	18	13CD3DC31, EEAG15631, 6AHDH67411, 4D7I412G11, D6BI135B21
63	4	56	9	1	6G1G0A41, HBI34EA1, C867ID11, CBE252D51, 97CGD7711, 7A7763C11, G4I79A231
96	4	87	24	1	C4H7E0ECE701GCH051E41, H5A492ICB92ABF3AD8A8251, C8548I94DBA8ACI539DGAH1, 42122C69DD71H8GE68F1431
120	4	110	20	2	3GIB2CF0F24F24E651, 6D75E887G83DAGA26D1, HDA9644B25IC6F9H411, DI16AD81C6FI0E4H741, 55EAA3DHG58D7E612711, 3CA6E96C97C2E3985121
126	4	115	18	1	E6F3G217B98122BB1, AD11AE3B7F84GHFF1, C3FBI671555E7D1I1, 9DE04FI420528A421, 53B4EDH98E3082031, 478F25744FF7A6B31,B1H4F26C914HDAH81
144	4	132	18	1	GABH3A343HAE98HC1, 7I1DD1C2H9DEE50G1, 4416G72C83I5CC4H1, 366B7DE6DFDH91411, H37GA18HB4159AI71, 95E253824427IBG411, GD5EGG5H38F1I64711, 8581I44EI8I5A45A11
152	4	139	19	1	AEECAAE54DFC6IC9B1, 9A4BDB6IA2E979GAB1, 71IIIGAHG5126BFGC1, GC8546C42714H31CI1, G9FD1FFI366186C521, DA332B38I62H5F1C211, BH5I46A268D7E1BB611, 91AC2D228FICBA47E11
19	5	15	19	1	15FGDC171CDGF51
30	5	24	10	1	1BDDB1, AC38032C1, 3B3F9FC621
40	5	33	10	1	1BDDB1, I9IBF4C1, BB3FFD7H1, FE3I6H1F21
50	5	42	10	1	2EF97331, EA88HF8F1, 92G27IG81, IDC1HF9C51, H8EG7IHF31
60	5	51	20	1	13AD5AEIIEA5DA31, 99HD10359B9H88IC1, 3AH24848AE29CB6B161
76	5	66	19	1	7741027E2637CBFF1, C4B3IA159B1IH27121, 2DDDH3I73DIBGC9AB1, B514655GG81AIECD511
80	5	69	20	1	I406H6ED24352H5I3B1, I5FB97HCBG3E8HEC4511, 4I1I64F907931E1GG31, AII673CDE3F74912CE1
100	5	88	20	1	GHII755F4E2EEGIG3B1, 2GF325422A4I64HIA61, 72CD9AB78ID10IDA721, F67DE771C4941H6C7F1, C90H6114B183625AI21
114	5	101	19	1	15FGDC171CDGF51, HCD2A3FF2572E2C6E1, 71H8H68D654BAI99111, 8HC4C6259GHB5E46211, CBCE0D4358D5A07961, GH36E95A7BCDBGD071
120	5	106	20	1	D822I5E6DFF51I11761, B8HA6H4A97699I53G1, I5FB97HCBG3E8HEC4511, BCD3FAF60E1EG0D1171, EF740425H432GEDFH81, 63HBA2EI9GI2IIDD151
133	5	118	19	1	EFCC11DHA6HF82171, 2E7B1B19ID06331D01, B6658C69D292H68GF1, 7H11G259C71F8A83A1, 3184BHEH776GF47E01, E8AC9I239B315I83F21, AE81F9C9H99G4HHH51
63	6	52	21	2	4254351F4BE81DE1, 8A9DBG5E41F86ICB8I421, 2EACE8B5434D0BCE6B41
98	6	84	49	1	E4HAF1275B27CDED6CF4F18HGDC5A9729GA712BGBB4HG4111, DC5ED5B2109HA4FAIF585I7EE2F306H9A365529AF94G1601
105	6	90	21	2	8A9DBG5E41F86ICB8I421, 2EACE8B5434D0BCE6B41, 8DD16C27D78ABD5IA6C1, DIIA999ABF4595127E61, CCAIGE18372C2DH6H5511
147	6	129	49	1	7G797F7E1GA2EDH66B979859FH4FC83AE8CE12279141E01, DICA1GDA64GEI92H348A712824 43975A4A30B8DAH33FH2G1, DC179GD20F5D8C15FFFD932CH42160DHGB0AC02G63371D31

22 and dimension 4 over $GF(19)$, the theoretical upper bound on the minimum distance is 19 whereas the actual minimum distance of a BKLC of this length and dimension is 18.

The following denotes the indexes used in Table 5:

Ba – Simeon Ball [21]

Be – Maximum distance separable code for $n < 21$ [17]

New – new codes presented in this paper

Gu – quasi-twisted code [25]

Unmarked entries can be obtained by puncturing and extending techniques

Table 5: Lower and upper bounds on minimum distances of linear codes over $GF(19)$

n	**k = 3**	**k = 4**	**k = 5**	**k = 6**	**n**	**k = 3**	**k = 4**	**k = 5**	**k = 6**

					76	70-71	68-69	66-69 New	63-68
					77	71-72	69-70	66-69	64-69
3	1				78	72	70-71	66-70	65-69
4	2	1			79	73	71-72	67-71	66-70
5	3	2	1		80	74	72-73 New	69-72 New	67-71
6	4	3	2	1	81	75	72-74	69-73	68-72
7	5	4	3	2	82	76	73-75	70-74	69-73
8	6	5	4	3	83	77	74-76	71-75	70-74
9	7	6	5	4	84	78	75-77	72-76	71-75 New
10	8	7	6	5	85	79	76-78	73-77	71-76
11	9	8	7	6	86	80	77-79	74-78	72-77
12	10	9	8	7	87	81 Ba	78-80	75-79	73-78
13	11	10	9	8	88	81-82	79-81	76-80	74-78
14	12	11	10	9	89	82-83	80-82	77-81	75-79
15	13	12	11	10	90	83-84	81-83	78-82	76-80
16	14	13	12	11	91	84-85	82-84	79-83	77-81
17	15	14	13	12	92	85-86	83-85	80-84	78-82
18	16	15	14	13	93	86-87	84-86	81-85	79-83
19	17	16	15	14	94	87-88	85-87	82-86	80-84
20	18 Be	17 Be	16 Be	15 Be	95	88-89	86-87	83-87	81-85
21	18	17-18	16-17	15-16	96	89-90	87-88 New	84-87	82-86
22	19	18-19	16-18	16-17	97	90	87-89	85-88	83-87
23	20	19	17-19	17-18	98	91	88-90	86-89	84-88 New
24	21	20	18-19	18-19 New	99	92	89-91	87-90	84-89
25	22	21	19-20	18-19	100	93	90-92	88-91 New	85-90
26	23	22	20-21	18-20	101	94	91-93	88-92	86-91
27	24	23	21-22	19-21	102	95	92-94	89-93	87-92
28	25	24 New	22-23	20-22	103	96	93-95	90-94	88-93
29	26	24-25	23-24	21-23	104	97	94-96	91-95	89-94
30	27	25-26	24-25 New	22-24	105	98 Ba	95-97	92-96	90-95 New
31	28 Ba	26-27	24-26	23-25	106	98-99	96-98	93-97	90-96
32	28-29	27-28	25-27	24-26	107	99-100	97-99	94-98	91-97
33	29-30	28-29	26-28	25-27	108	100-101	98-100	95-99	92-98
34	30-31	29-30	27-29	26-28	109	101-102	99-101	96-100	93-99
35	31-32	30-31	28-30	27-29	110	102-103	100-102	97-101	94-100
36	32-33	31-32 New	29-31	28-30 New	111	103-104	101-103	98-102	95-101
37	33-34	31-33	30-32	28-31	112	104-105	102-104	99-103	96-102
38	34-35	32-34	31-33	29-32	113	105-106	103-105	100-104	97-103
39	35-36	33-35	32-34	30-33	114	106-107	104-106	101-105 New	98-104
40	36	34-36	33-35 New	31-34	115	107-108	105-106	101-106	99-105
41	37	35-37	33-36	32-35	116	108	106-107	102-106	100-106
42	38	36-37	34-37	33-36 New	117	109	107-108	103-107	101-106
43	39	37-38	35-37	33-37	118	110	108-109	104-108	102-107
44	40	38-39	36-38	34-38	119	111	109-110	105-109	103-108
45	41	39-40	37-39	35-38	120	112	110-111 New	106-110 New	104-109 New
46	42	40-41	38-40	36-39	121	113	110-112	106-111	104-110
47	43	41-42	39-41	37-40	122	114	111-113	107-112	105-111
48	44	42-43	40-42	38-41	123	115	112-114	108-113	106-112
49	45	43-44	41-43	39-42	124	116	113-115	109-114	107-113
50	46	44-45 New	42-44 New	40-43	125	117	114-116	110-115	108-114
51	47	44-46	42-45	41-44	126	118 Ba	115-117 New	111-116	109-115 New
52	48 Ba	45-47	43-46	42-45	127	118-119	115-118	112-117	109-116

53	48-49	46-48	44-47	43-46	128	119-120	116-119	113-118	110-117
54	49-50	47-49	45-48	44-47 New	129	120-121	117-120	114-119	111-118
55	50-51	48-50	46-49	44-48	130	121-122	118-121	115-120	112-119
56	51-52	49-51	47-50	45-49	131	122-123	119-122	116-121	113-120
57	52-53	50-52	48-51	46-50	132	123 -124	120-123	117-122	114-121
58	53-54	51-53	49-52	47-51	133	124-125	121-124	118-123 New	115-122
59	54	52-53	50-53	48-52	134	125-126	122-125	118-123	116-123
60	55	53-54	51-53 New	49-53	135	126	123-125	119-124	117-123
61	56	54-55	51-54	50-53	136	127	124-126	120-125	118-124
62	57	55-56	52-55	51-54	137	128	125-127	121-126	119-125
63	58	55-57	53-56	52-55 New	138	129	126-128	122-127	120-126
64	59	56-57 New	54-57	52-56	139	130	127-129	123-128	121-127
65	60	57-59	55-58	53-57	140	131	128-130	124-129	122-128
66	61	58-60	56-59	54-58	141	132	129-131	125-130	123-129
67	62	59-61	57-60	55-59	142	133	130-132	126-131	124-130
68	63 Ba	60-62	58-61 New	56-60	143	134	131-133	127-132	125-131
69	63-64	61-63	59-62	57-61	144	135	132-134 New	128-133	126-132
70	64-65	62-64	60-63	58-62	145	136	132-135	129-134	127-133
71	65-66	63-65	61-64	59-63	146	137	133-136	130-135	128-134
72	66-67	64-66	62-65	60-64 New	147	138 Ba	134-137	131-136	129-135 New
73	67-68	65-67	63-66	60-65	148	138-139	135-138	132-137	129-136
74	68-69	66-68	64-67	61-66	149	139-140	136-139	133-138	130-137
75	69-70	67-69	65-68	62-67	150	140-141	137-140 New	134-138 New	131-138 New

5. Conclusion and future directions

This work introduces a partial database of BKLCs over the finite field $GF(19)$. The database covers codes for lengths up to 150 and dimensions up to 6. For these small dimensions, we searched for codes with good parameters among the class of QT codes, which includes cyclic, constacyclic, and QC codes special cases. This search was quite effective. As the code dimension gets larger, the search for good codes becomes computationally taxing. Although the class of QT codes remains promising, it is necessary to employ other types of codes as well. Moreover, other search methods and approaches are needed to address the challenging problem of finding codes with better parameters in higher dimensions. While there is no limit to the variety of methods that can be used to make progress, some possibilities include heuristic search algorithms, genetic search algorithms, and making use of AI. Finding codes with best possible parameters is one of the major problems of coding theory with many open cases, and it invites ingenuity and creativity of researchers to come up with new approaches. No single method will solve all open cases of the problem.

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