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On distance spectra, energies and Wiener index of non-commuting conjugacy class graphs

Research Article

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Abstract: The non-commuting conjugacy class graph (abbreviated as NCCC-graph) of a finite non-abelian group H is a simple undirected graph whose vertex set is the set of conjugacy classes of non-central elements of H and two vertices, a^H and b^H are adjacent if $a'b' \neq b'a'$ for all $a' \in a^H$ and $b' \in b^H$. In this paper, we compute distance spectrum, distance Laplacian spectrum, distance signless Laplacian spectrum along with their respective energies and Wiener index of NCCC-graphs of H when the central quotient of H is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (for any prime p) or D_{2n} (for any integer $n \geq 3$). As a consequence, we compute various distance spectra, energies and Wiener index of NCCC-graphs of the dihedral group, dicyclic group, semidihedral group along with the groups $U_{(n,m)}$, U_{6n} and V_{8n}. Thus we obtain sequences of positive integers that can be realized as Wiener index of NCCC-graphs of certain groups. In particular, we solve Inverse Wiener index Problem for NCCC-graphs of groups when n is a perfect square. We further characterize the above-mentioned groups such that their NCCC-graphs are D-integral, DL-integral and DQ-integral. We also compare various distance energies of NCCC-graphs of the above mentioned groups and characterize those groups subject to the inequalities involving various distance energies.

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1. Introduction

Let H be a finite non-abelian group with center $Z(H)$. The commuting conjugacy class graph (CCC-graph) of H is a simple undirected graph whose vertex set consists of the conjugacy classes of non-central elements of H and two vertices, a^H and b^H, are adjacent if there exist elements $a' \in a^H$ and

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$b' \in b^H$ such that $a'b' = b'a'$. The study of CCC-graphs of groups was initiated by Herzog, Longobardi, and Maj [25] in 2009. In 2016, Mohammadian et al. [31] characterized finite groups whose CCC-graphs are triangle-free. Subsequently, Salahshour and Ashrafi [37, 38] investigated the structures of CCC-graphs for various families of finite CA-groups. Salahshour [36] also described CCC-graphs for groups whose central quotient is isomorphic to a dihedral group. Using those structures of CCC-graphs, Bhowal and Nath [9, 10] have computed spectrum, Laplacian spectrum, signless Laplacian spectrum and their corresponding energies. Bhowal and Nath also obtained various groups such that CCC-graphs are hyperenergetic, L-hyperenergetic and Q-hyperenergetic. Recently, Jannat and Nath [27] have computed various CN-spectra and CN-energies of of CCC-graphs of groups and obtained finite non-abelian groups such that their CCC-graphs are CN-hyperenergetic, CNL-hyperenergetic and CNSL-hyperenergetic. All these results can also be found in the survey article [12] written by Cameron et al. Spectrum and Laplacian spectrum of conjugacy super commuting graph (an extended version of CCC-graph) of dihedral groups and generalized quaternion groups have been computed in [14]. In this paper, we consider the non-commuting conjugay class graph (NCCC-graph) of H which is the complement of CCC-graph of H. Thus, NCCC-graph of H is a simple undirected graph whose vertex set is the set of conjugacy classes of non-central elements of H and two vertices a^H and b^H are adjacent if $a'b' \neq b'a'$ for all $a' \in a^H$ and $b' \in b^H$. We write Γ_H to denote the NCCC-graph of H.

Let $\mathcal{G}$ be a connected graph with vertex set $V(\mathcal{G}) = \{v_1, v_2, \ldots, v_n\}$ and distance matrix $\mathcal{D}(\mathcal{G})$. Define $\mathcal{T}(\mathcal{G})$ as a diagonal matrix whose i-th diagonal entry $\mathcal{T}(\mathcal{G})_{i,i}$ is the transmission of vertex v_i in $\mathcal{G}$ given by $\mathcal{T}(\mathcal{G})_{i,i} = \sum_{j=1}^{n} d_{ij}$, where d_{ij} is the (i,j)-th element of $\mathcal{D}(\mathcal{G})$. Note that $d_{ij} = d(v_i, v_j)$ is the distance between v_i and v_j. The distance Laplacian matrix $DL(\mathcal{G})$ and the distance signless Laplacian matrix $DQ(\mathcal{G})$ are defined as follows:

$$DL(\mathcal{G}) := \mathcal{T}(\mathcal{G}) - \mathcal{D}(\mathcal{G}) \text{ and } DQ(\mathcal{G}) := \mathcal{T}(\mathcal{G}) + \mathcal{D}(\mathcal{G}).$$

The distance spectrum D-spec($\mathcal{G}$), distance Laplacian spectrum DL-spec($\mathcal{G}$) and distance signless Laplacian spectrum DQ-spec($\mathcal{G}$) are the sets of eigenvalues of $\mathcal{D}(\mathcal{G})$, $DL(\mathcal{G})$ and $DQ(\mathcal{G})$ respectively, each with their corresponding multiplicities. Specifically, we write D-spec($\mathcal{G}$) $= \{[\alpha_1]^{a_1}, [\alpha_2]^{a_2}, \ldots, [\alpha_l]^{a_l}\}$, DL-spec($\mathcal{G}$) $= \{[\beta_1]^{b_1}, [\beta_2]^{b_2}, \ldots, [\beta_m]^{b_m}\}$ and DQ-spec($\mathcal{G}$) $= \{[\gamma_1]^{c_1}, [\gamma_2]^{c_2}, \ldots, [\gamma_q]^{c_q}\}$, where $\alpha_1, \alpha_2, \ldots, \alpha_l$ are the eigenvalues of $\mathcal{D}(\mathcal{G})$ with multiplicities $a_1, a_2, \ldots, a_l$; $\beta_1, \beta_2, \ldots, \beta_m$ are the eigenvalues of $DL(\mathcal{G})$ with multiplicities $b_1, b_2, \ldots, b_m$; $\gamma_1, \gamma_2, \ldots, \gamma_q$ are the eigenvalues of $DQ(\mathcal{G})$ with multiplicities $c_1, c_2, \ldots, c_q$ respectively. A connected graph $\mathcal{G}$ is called distance integral (D-integral), distance Laplacian integral (DL-integral) or distance signless Laplacian integral (DQ-integral) if D-spec($\mathcal{G}$), DL-spec($\mathcal{G}$) or DQ-spec($\mathcal{G}$), respectively, consist solely of integer eigenvalues. The study of D-spec($\mathcal{G}$) was pioneered by Indulal and Gutman [26] in 2008, while the concepts of DL-spec($\mathcal{G}$) and DQ-spec($\mathcal{G}$) were introduced by Aouchiche and Hansen [4] in 2013. For further reading on distance (Laplacian) spectra, we refer to [5, 7] and the references therein. The study of integral graphs began with the question, "Which graphs have integral spectra?" posed by Harary and Schwenk [24] in 1974 (also see [8] for more information). Ahmadi et al. [1] highlighted the significance of integral graphs in designing network topologies for perfect state transfer networks. The distance matrices also have a wide range of applications across various fields, including the design of communication networks, network flow algorithms, graph embedding theory and even in areas such as molecular stability, branching and model boiling points of an alkane, psychology, phylogenetics, software compression, analysis of internet infrastructures, modeling of traffic and social networks etc. as noted in [5]. However, we shall not address any of these applications in the present work.

The Wiener index of $\mathcal{G}$, denoted by $W(\mathcal{G})$, is defined as $W(\mathcal{G}) = \frac{1}{2}\sum_{1 \leq i,j \leq n} d(v_i, v_j)$. This is the oldest topological index of a graph originated from the work of Wiener [44] in 1947. For various results on Wiener index we refer to [17, 47] and the references there in. Recently in [50], Brouwer type relation between sum of distance Laplacian eigenvalues and Wiener index of connected graphs of diameter one and two is obtained. One interesting problem regarding Wiener index is the Inverse Wiener index Problem: given any positive integer n find a graph $\mathcal{G}$ from a prescribed class such that $W(\mathcal{G}) = n$. Considering the family of all graphs Gutman et al. [23] solved the Inverse Wiener index Problem for $n \neq 2, 5$. The Inverse Wiener index Problem is also solved for bipartite graphs, trees etc. with some exceptional values of n (see [43] for details). However, Inverse Wiener index Problem is not solved for graphs defined on groups, in particular for NCCC-graphs. In this paper, we consider NCCC-graphs of groups and solved Inverse Wiener index Problem when n is a perfect square.

Topological indices of a graph are numerical quantities derived from the graph. In Mathematical Chemistry, Topological indices play a crucial role in quantitative structure-property relation (QSPR) and quantitative structure-activity relation (QSAR) studies. Among the degree based topological indices, first Zagreb index and second Zagreb index were the oldest and these were introduced by Gutman and Trinajstić [22] in 1972. Later on, various types of Zagreb indices were introduced. Recent results on these indices along with their applications in Mathematical Chemistry can be found in [3, 32–35].

It is worth mentioning that the Wiener index of $\mathcal{G}$ plays a major role in computing various distance energies of $\mathcal{G}$. In analogy to energy $\mathrm{E}(\mathcal{G})$, Laplacian energy $\mathrm{LE}(\mathcal{G})$ and signless Laplacian energy $\mathrm{LE}^+(\mathcal{G})$ of a graph $\mathcal{G}$ Indulal et al. [26], Gutman et al. [48] and Das et al. [15] introduced distance energy $\mathrm{E_D}(\mathcal{G})$, distance Laplacian energy $\mathrm{E_{DL}}(\mathcal{G})$ and distance signless Laplacian energy $\mathrm{E_{DQ}}(\mathcal{G})$ of $\mathcal{G}$ as given below:

$$\mathrm{E_D}(\mathcal{G}) = \sum_{\alpha \in \text{D-spec}(\mathcal{G})} |\alpha|, \qquad \mathrm{E_{DL}}(\mathcal{G}) = \sum_{\beta \in \text{DL-spec}(\mathcal{G})} \left| \beta - \frac{tr(DL(\mathcal{G}))}{|V(\mathcal{G}|} \right|$$

and

$$\mathrm{E_{DQ}}(\mathcal{G}) = \sum_{\gamma \in \text{DQ-spec}(\mathcal{G})} \left| \gamma - \frac{tr(DQ(\mathcal{G}))}{|V(\mathcal{G}|} \right|.$$

Various results on $\mathrm{E_D}(\mathcal{G})$ have been extensively studied by several mathematicians (for example see [5, 26, 41, 45, 49]). Further, several bounds of $\mathrm{E_D}(\mathcal{G})$, $\mathrm{E_{DL}}(\mathcal{G})$ and $\mathrm{E_{DQ}}(\mathcal{G})$ and relations among these three energies have been explored in [15, 16, 48] and the references therein.

In Section 2, we compute distance spectrum, distance Laplacian spectrum, distance signless Laplacian spectrum and Wiener index of Γ_H for the groups when $\frac{H}{Z(H)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (for any prime p) or D_{2n} (for any integer $n \geq 3$). As a consequence, we compute the above mentioned graph parameters for NCCC-graphs of the dihedral group $D_{2n} = \langle x, y : x^n = y^2 = 1, yxy^{-1} = x^{-1} \rangle$ (for $n \geq 3$), the dicyclic group $T_{4n} = \langle x, y : x^{2n} = 1, x^n = y^2, y^{-1}xy = x^{-1} \rangle$ (for $n \geq 2$), the semidihedral group $SD_{8n} = \langle x, y : x^{4n} = y^2 = 1, yxy^{-1} = x^{2n-1} \rangle$ (for $n \geq 2$) and the groups $U_{(n,m)} = \langle x, y : x^{2n} = y^m = 1, x^{-1}yx = y^{-1} \rangle$ (for $m \geq 3$ and $n \geq 2$), $U_{6n} = \langle x, y : x^{2n} = y^3 = 1, x^{-1}yx = y^{-1} \rangle$ (for $n \geq 2$) and $V_{8n} = \langle x, y : x^{2n} = y^4 = 1, yx = x^{-1}y^{-1}, y^{-1}x = x^{-1}y \rangle$ (for $n \geq 2$). It is observed that any perfect square can be realized as Wiener index of NCCC-graphs of certain dihedral groups. We also characterize the above mentioned groups such that their NCCC-graphs are D-integral, DL-integral and DQ-integral. In Section 3, we compute distance energy, distance Laplacian energy and distance signless Laplacian energy of NCCC-graphs of the above mentioned groups. Further, In Section 4 we compare various distance energies of NCCC-graphs of groups and characterize the above-mentioned groups subject to the inequalities involving various distance energies. Comparison of various graph energies became interesting due to the E-LE conjecture: Is $\mathrm{E}(\mathcal{G}) \leq \mathrm{LE}(\mathcal{G})$? (see [21, 29, 42]) which is eventually proved to be false. Das et. al. [15] mentioned the following problem comparing $\mathrm{LE}(\mathcal{G})$ and $\mathrm{LE}^+(\mathcal{G})$.

Problem 1.1. [15, Problem 1] *Characterize all the graphs for which* $\mathrm{LE}(\mathcal{G}) > \mathrm{LE}^+(\mathcal{G})$, $\mathrm{LE}(\mathcal{G}) < \mathrm{LE}^+(\mathcal{G})$ *and* $\mathrm{LE}(\mathcal{G}) = \mathrm{LE}^+(\mathcal{G})$.

The energy, Laplacian energy and signless Laplacian energy of graphs defined on groups such as commuting graph, non-commuting graph and CCC-graph were compared in [9, 11, 18–20, 40]. Das et. al. [15] also posed the following problems.

Problem 1.2. [15, Problem 3] *Characterize all the graphs for which* $\mathrm{E_{DL}}(\mathcal{G}) = \mathrm{E_{DQ}}(\mathcal{G})$.

Problem 1.3. [15, Problem 4] *Is there any connected graph* $\mathcal{G}(\ncong K_n)$ *such that* $\mathrm{E}(\mathcal{G}) = \mathrm{LE}(\mathcal{G}) = \mathrm{LE}^+(\mathcal{G}) = \mathrm{E_D}(\mathcal{G}) = \mathrm{E_{DL}}(\mathcal{G}) = \mathrm{E_{DQ}}(\mathcal{G})$*?*

In Section 3 and 4, we consider Problem 1.2 and Problem 1.3 for NCCC-graphs of the above mentioned groups and obtain graphs satisfying the equalities in Problem 1.2 and Problem 1.3.

2. Distance spectra and Wiener index

In this section, we compute distance spectrum, distance Laplacian spectrum, distance signless Laplacian spectrum and Wiener index of Γ_H for the groups when $\frac{H}{Z(H)}$ is isomorphic to

(a). $\mathbb{Z}_p \times \mathbb{Z}_p$, where p is any prime.

(b). D_{2n}, where $n \geq 3$ is any integer.

As consequences, we get various distance spectra and Wiener index of Γ_H if $H = D_{2n}$, T_{4n}, SD_{8n}, $U_{(n,m)}$, U_{6n} and V_{8n}. The following simple-minded result is very useful in computing Wiener index of any finite graph. However, this relation was neglected while computing Wiener index of various graphs (see [2, 17, 30, 39, 47]).

Lemma 2.1. *Let $\mathcal{G}$ be any graph having n vertices. Then*

$$W(\mathcal{G}) = \frac{1}{2} \sum_{\beta \in \text{DL-spec}(\mathcal{G})} \beta = \frac{1}{2} \sum_{\gamma \in \text{DQ-spec}(\mathcal{G})} \gamma.$$

Proof. From the definitions of $DL(\mathcal{G})$ and $DQ(\mathcal{G})$ we have $tr(DL(\mathcal{G})) = tr(\mathcal{T}(\mathcal{G})) = tr(DQ(\mathcal{G}))$. Also,

$$tr(\mathcal{T}(\mathcal{G})) = \sum_{1 \leq i,j \leq n} d_{ij} = \sum_{1 \leq i,j \leq n} d(v_i, v_j).$$

Therefore, $tr(\mathcal{T}(\mathcal{G})) = 2W(\mathcal{G})$ and so

$$tr(DL(\mathcal{G})) = tr(DQ(\mathcal{G})) = 2W(\mathcal{G}). \quad (1)$$

Since trace of a square matrix is equal to the sum of its eigenvalues we have

$$\sum_{\beta \in \text{DL-spec}(\mathcal{G})} \beta = tr(DL(\mathcal{G})) = tr(DQ(\mathcal{G})) = \sum_{\gamma \in \text{DQ-spec}(\mathcal{G})} \gamma.$$

Hence, the result follows. □

Note that the NCCC-graphs of the groups considered in this section are complete k-partite graphs (see [36, 37]). Therefore, the following result is useful in computing various distance spectra.

Lemma 2.2. *Let $\mathcal{G} = K_{n_1, n_2, \ldots, n_k}$ be a complete k-partite graph, where $2 \leq k \leq |V(\mathcal{G})| - 1$. Then*

(a). [28, Lemma 2.5] *the characteristic polynomial of $D(\mathcal{G})$ is given by*

$$\text{Ch}_D(x) = (x+2)^{|V(\mathcal{G})|-k} \left(\prod_{i=1}^{k} (x - n_i + 2) - \sum_{i=1}^{k} n_i \prod_{j=1, j \neq i}^{k} (x - n_j + 2) \right).$$

(b). [28, Lemma 2.8] *the characteristic polynomial of $DL(\mathcal{G})$ is given by*

$$\text{Ch}_{DL}(x) = x(x - |V(\mathcal{G})|)^{k-1} \prod_{i=1}^{k} (x - |V(\mathcal{G})| - n_i)^{n_i - 1}.$$

(c). [28, Lemma 2.12] *the characteristic polynomial $\text{Ch}_{DQ}(x)$ of $DQ(\mathcal{G})$ is given by*

$$\prod_{i=1}^{k} (x - |V(\mathcal{G})| - n_i + 4)^{n_i - 1} \left(\prod_{i=1}^{k} (x - |V(\mathcal{G})| - 2n_i + 4) - \sum_{i=1}^{k} n_i \prod_{j=1, j \neq i}^{k} (x - |V(\mathcal{G})| - 2n_j + 4) \right).$$

The following theorem gives various distance spectra and Wiener index of NCCC-graphs of finite groups whose central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$.

Theorem 2.3. *Let H be a finite non-abelian group such that $\frac{H}{Z(H)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is any prime. If $n = \frac{(p-1)z}{p}$, where $z = |Z(H)|$, then*

$$\text{D-spec}(\Gamma_H) = \{[-2]^{(n-1)(p+1)}, [n-2]^p, [np+2n-2]^1\},$$

$$\text{DL-spec}(\Gamma_H) = \{[0]^1, [n(p+1)]^p, [n(p+1)+n]^{(p+1)(n-1)}\},$$

$$\text{DQ-spec}(\Gamma_H) = \{[np+2n-4]^{(p+1)(n-1)}, [np+3n-4]^p, [2np+4n-4]^1\}$$

and $W(\Gamma_H) = \frac{n(p+1)(n(p+2)-2)}{2}$.

Proof. By [37, Theorem 3.1], we have $\Gamma_H = K_{n_1,n_2,\ldots,n_{p+1}}$, where $n_1 = n_2 = \cdots = n_{p+1} = n = \frac{(p-1)z}{p}$. Here, $|V(H)| = (p+1)n$. Therefore, by Lemma 2.2(a), we have

$$\begin{aligned}\text{Ch}_D(x) &= (x+2)^{(n-1)(p+1)}\left(\prod_{i=1}^{p+1}(x-n_i+2) - \sum_{i=1}^{p+1} n_i \prod_{j=1, j\neq i}^{p+1}(x-n_j+2)\right)\\ &= (x+2)^{(n-1)(p+1)}\left((x-n+2)^p(x-np-2n+2)\right).\end{aligned}$$

Hence, $\text{D-spec}(\Gamma_H) = \left\{[-2]^{(n-1)(p+1)}, [n-2]^p, [np+2n-2]^1\right\}$.

By Lemma 2.2(b), we have

$$\begin{aligned}\text{Ch}_{DL}(x) &= x(x-n(p+1))^{(p+1)-1}\prod_{i=1}^{p+1}(x-(p+1)n-n_i)^{n_i-1}\\ &= x(x-n(p+1))^p(x-n(p+1)-n)^{(p+1)(n-1)}.\end{aligned}$$

Therefore, $\text{DL-spec}(\Gamma_H) = \left\{[0]^1, [n(p+1)]^p, [n(p+1)+n]^{(p+1)(n-1)}\right\}$.

By Lemma 2.2(c), we have

$$\begin{aligned}\text{Ch}_{DQ}(x) = \prod_{i=1}^{p+1}(x-n(p+1)-n_i+4)^{n_i-1}&\left(\prod_{i=1}^{p+1}(x-n(p+1)-2n_i+4) - \right.\\ &\left.\sum_{i=1}^{p+1} n_i \prod_{j=1, j\neq i}^{p+1}(x-n(p+1)-2n_j+4)\right)\\ = (x-np-2n+4)^{(p+1)(n-1)}&(x-np-3n+4)^p(x-2np-4n+4).\end{aligned}$$

Therefore, $\text{DQ-spec}(\Gamma_H) = \left\{[np+2n-4]^{(p+1)(n-1)}, [np+3n-4]^p, [2np+4n-4]^1\right\}$. The expression for $W(\Gamma_H)$ follows from Lemma 2.1. ☐

If H is a non-abelian group of order p^n with $|Z(H)| = p^{n-2}$, where p is prime and $n \geq 3$, then $\frac{H}{Z(H)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Therefore, we have the following corollary.

Corollary 2.4. *Let H be a non-abelian group of order p^n with $|Z(H)| = p^{n-2}$, where p is prime and $n \geq 3$. Then*

$$\text{D-spec}(\Gamma_H) = \left\{[-2]^{(p+1)\left((p-1)p^{n-3}-1\right)}, [(p-1)p^{n-3}-2]^p, [2(p-1)p^{n-3}+(p-1)p^{n-2}-2]^1\right\},$$

$$\text{DL-spec}(\Gamma_H) = \left\{[0]^1, [\left(p^2-1\right)p^{n-3}]^p, [\left(p^2+p-2\right)p^{n-3}]^{(p+1)\left((p-1)p^{n-3}-1\right)}\right\}$$

$$\text{DQ-spec}(\Gamma_H) = \Big\{[-2p^{n-3}+p^{n-2}+p^{n-1}-4]^{(p+1)\left((p-1)p^{n-3}-1\right)},$$
$$[-3p^{n-3}+2p^{n-2}+p^{n-1}-4]^p, [2\left(p^2+p-2\right)p^{n-3}-4]^1\Big\}.$$

and $W(\Gamma_H) = \frac{(p-1)(p+1)p^{n-3}\left((p-1)(p+2)p^{n-3}-2\right)}{2}$.

The following theorem gives various distance spectra and Wiener index of NCCC-graphs of finite groups whose central quotient is isomorphic to a dihedral group.

Theorem 2.5. *Let H be a finite non-abelian group with $|Z(H)| = z$ and $\frac{H}{Z(H)} \cong D_{2n}$, (where $n \geq 3$).*
(a). *If n is even then*

$$\text{D-spec}(\Gamma_H) = \Big\{[-2]^{\frac{1}{2}(n+1)z-3}, [\tfrac{z}{2}-2]^1, \big[\tfrac{1}{4}(-\sqrt{4n^2-12n+17}z+2nz+z-8)\big]^1,$$
$$(\sqrt{4n^2-12n+17}z+2nz+z-8)\big]^1\Big\},$$

$$\text{DL-spec}(\Gamma_H) = \left\{[0]^1, \left[\tfrac{(n+1)z}{2}\right]^2, [nz]^{\frac{(n-1)z}{2}-1}, \left[\tfrac{(n+2)z}{2}\right]^{z-2}\right\},$$

$$\text{DQ-spec}(\Gamma_H) = \Big\{[nz-4]^{\frac{(n-1)z}{2}-1}, \left[\tfrac{(n+2)z}{2}-4\right]^{z-2}, \left[\tfrac{(n+3)z}{2}-4\right]^1,$$
$$\big[\tfrac{1}{4}\left(-\sqrt{9n^2-34n+41}z+5nz+3z-16\right)\big]^1, \big[\tfrac{1}{4}\left(\sqrt{9n^2-34n+41}z+5nz+3z-16\right)\big]^1\Big\}$$

and $W(\Gamma_H) = \frac{1}{4}z\left(n^2z-2n+2z-2\right)$.

(b). *If n is odd then*

$$\text{D-spec}(\Gamma_H) = \Big\{[-2]^{\frac{(n+1)z}{2}-2}, \big[\tfrac{1}{2}\left(-\sqrt{n^2-4n+7}z+nz+z-4\right)\big]^1,$$
$$\big[\tfrac{1}{2}\left(\sqrt{n^2-4n+7}z+nz+z-4\right)\big]^1\Big\},$$

$$\text{DL-spec}(\Gamma_H) = \left\{[0]^1, \left[\tfrac{(n+1)z}{2}\right]^1, [nz]^{\frac{(n-1)z}{2}-1}, \left[\tfrac{(n+3)z}{2}\right]^{z-1}\right\},$$

$$\text{DQ-spec}(\Gamma_H) = \Big\{[nz-4]^{\frac{(n-1)z}{2}-1}, \left[\tfrac{(n+3)z}{2}-4\right]^{z-1}, \big[\tfrac{1}{4}\left(-\sqrt{9n^2-46n+73}z+5nz+5z-16\right)\big]^1,$$
$$\big[\tfrac{1}{4}\left(\sqrt{9n^2-46n+73}z+5nz+5z-16\right)\big]^1\Big\}$$

and $W(\Gamma_H) = \frac{1}{4}z\left(n^2z-2n+3z-2\right)$.

Proof. (a). If n is even then By [36, Theorem 1.2], we have $\Gamma_H = K_{\frac{(n-1)z}{2},\frac{z}{2},\frac{z}{2}}$. Here, $|V(\Gamma_H)| = \frac{(n+1)z}{2}$.

Using Lemma 2.2(a), we get

$$\begin{aligned}
\text{Ch}_D(x) &= (x+2)^{\frac{(n+1)z}{2}-3}\left(\prod_{i=1}^{3}(x-n_i+2) - \sum_{i=1}^{3} n_i \prod_{j=1, j\neq i}^{3}(x-n_j+2)\right)\\
&= (x+2)^{\frac{1}{2}(n+1)z-3}(x-\frac{z}{2}+2)\Bigg((x-\frac{(n-1)z}{2}+2)(x-\frac{z}{2}+2)\\
&\quad -\frac{(n-1)z}{2}(x-\frac{z}{2}+2) - z(x-\frac{(n-1)z}{2}+2)\Bigg).
\end{aligned}$$

Therefore, D-spec$(\Gamma_H) = \Big\{[-2]^{\frac{1}{2}(n+1)z-3}, [\frac{z}{2}-2]^1, \big[\frac{1}{4}\left(2nz+z-8+z\sqrt{4n^2-12n+17}\right)\big]^1,$ $\big[\frac{1}{4}\left(2nz+z-8-z\sqrt{4n^2-12n+17}\right)\big]^1\Big\}$.

Using Lemma 2.2(b), we get

$$\begin{aligned}\text{Ch}_{DL}(x) &= x\left(x-\frac{(n+1)z}{2}\right)^{3-1}\prod_{i=1}^{3}\left(x-\frac{(n+1)z}{2}-n_i\right)^{n_i-1}\\ &= x\left(x-\frac{(n+1)z}{2}\right)^{2}(x-nz)^{\frac{(n-1)z}{2}-1}\left(x-\frac{(n+2)z}{2}\right)^{z-2}.\end{aligned}$$

Therefore, DL-spec$(\Gamma_H) = \Big\{[0]^1, [\frac{(n+1)z}{2}]^2, [nz]^{\frac{(n-1)z}{2}-1}, [\frac{(n+2)z}{2}]^{z-2}\Big\}$.

Using Lemma 2.2(c), we get

$$\begin{aligned}\text{Ch}_{DQ}(x) &= \prod_{i=1}^{3}\left(x-\frac{(n+1)z}{2}-n_i+4\right)^{n_i-1}\left(\prod_{i=1}^{3}\left(x-\frac{(n+1)z}{2}-2n_i+4\right)\right.\\ &\qquad\left.-\sum_{i=1}^{3}n_i\prod_{j=1,j\neq i}^{3}\left(x-\frac{(n+1)z}{2}-2n_j+4\right)\right)\\ &= (x-nz+4)^{\frac{(n-1)z}{2}-1}\left(x-\frac{(n+2)z}{2}+4\right)^{z-2}\left(-\frac{1}{2}(n-1)z\left(-\frac{1}{2}(n+1)z+x-z+4\right)^2\right.\\ &\qquad+\left(-\frac{1}{2}(n+1)z-(n-1)z+x+4\right)\left(-\frac{1}{2}(n+1)z+x-z+4\right)^2\\ &\qquad\left.-z\left(-\frac{1}{2}(n+1)z-(n-1)z+x+4\right)\left(-\frac{1}{2}(n+1)z+x-z+4\right)\right).\end{aligned}$$

Therefore, DQ-spec$(\Gamma_H) = \Big\{[nz-4]^{\frac{(n-1)z}{2}-1}, [\frac{(n+2)z}{2}-4]^{z-2}, [\frac{(n+3)z}{2}-4]^1,$ $\big[\frac{1}{4}\left(5nz+3z-16-z\sqrt{9n^2-34n+41}\right)\big]^1, \big[\frac{1}{4}\left(5nz+3z-16+z\sqrt{9n^2-34n+41}\right)\big]^1\Big\}$. The expression for $W(\Gamma_H)$ follows from Lemma 2.1.

(b). If n is odd then by [36, Theorem 1.2], we have $\Gamma_H = K_{\frac{(n-1)z}{2},z}$. Here, $|V(\Gamma_H)| = \frac{(n+1)z}{2}$.

Using Lemma 2.2(a), we get

$$\begin{aligned}\text{Ch}_{D}(x) &= (x+2)^{\frac{(n+1)z}{2}-2}\left(\prod_{i=1}^{2}(x-n_i+2)-\sum_{i=1}^{2}n_i\prod_{j=1,j\neq i}^{3}(x-n_j+2)\right)\\ &= (x+2)^{\frac{(n+1)z}{2}-2}\left(\left(x-\frac{(n-1)z}{2}+2\right)(x-z+2)-\frac{(n-1)z}{2}(x-z+2)-z\left(x-\frac{(n-1)z}{2}+2\right)\right).\end{aligned}$$

Therefore, D-spec$(\Gamma_H) = \Big\{[-2]^{\frac{(n+1)z}{2}-2}, [\frac{1}{2}(nz+z-4-z\sqrt{n^2-4n+7})]^1, [\frac{1}{2}(nz+z-4+z\sqrt{n^2-4n+7})]^1\Big\}$.

Using Lemma 2.2(b), we get

$$\begin{aligned}\text{Ch}_{DL}(x) &= x\left(x-\frac{(n+1)z}{2}\right)^{2-1}\prod_{i=1}^{2}\left(x-\frac{(n+1)z}{2}-n_i\right)^{n_i-1}\\ &= x\left(x-\frac{(n+1)z}{2}\right)(x-nz)^{\frac{(n-1)z}{2}-1}\left(x-\frac{(n+3)z}{2}\right)^{z-1}.\end{aligned}$$

Therefore, DL-spec$(\Gamma_H) = \left\{[0]^1, \left[\frac{(n+1)z}{2}\right]^1, [nz]^{\frac{(n-1)z}{2}-1}, \left[\frac{(n+3)z}{2}\right]^{z-1}\right\}$.

Using Lemma 2.2(c), we get

$$\begin{aligned}\mathrm{Ch}_{DQ}(x) = \prod_{i=1}^{2}\left(x - \frac{(n+1)z}{2} - n_i + 4\right)^{n_i-1}\Bigg(\prod_{i=1}^{2}\left(x - \frac{(n+1)z}{2} - 2n_i + 4\right) \\ - \sum_{i=1}^{2} n_i \prod_{j=1, j\neq i}^{2}\left(x - \frac{(n+1)z}{2} - 2n_j + 4\right)\Bigg) \\ = (x - nz + 4)^{\frac{(n-1)z}{2}-1}\left(x - \frac{(n+3)z}{2} + 4\right)^{z-1}\Bigg(\left(x - \frac{(3n-1)z}{2} + 4\right) \\ \left(x - \frac{(n+5)z}{2} + 4\right) - \frac{(n-1)z}{2}\left(x - \frac{(n+5)z}{2} + 4\right) - z\left(x - \frac{(3n-1)z}{2} + 4\right)\Bigg).\end{aligned}$$

Therefore, DQ-spec$(\Gamma_H) = \Big\{[nz-4]^{\frac{(n-1)z}{2}-1}, \left[\frac{(n+3)z}{2} - 4\right]^{z-1}, \left[\frac{1}{4}(5nz + 5z - 16 - z\sqrt{9n^2 - 46n + 73})\right]^1,$ $\left[\frac{1}{4}(5nz + 5z - 16 + z\sqrt{9n^2 - 46n + 73})\right]^1\Big\}$. The expression for $W(\Gamma_H)$ follows from Lemma 2.1. □

Corollary 2.6. *Let H be the dihedral group D_{2n}, where $n \geq 3$.*

(a). *If n is odd then*

D-spec$(\Gamma_H) = \left\{[-2]^{\frac{n-3}{2}}, \left[\frac{1}{2}\left(-\sqrt{n^2-4n+7} + n - 3\right)\right]^1, \left[\frac{1}{2}\left(\sqrt{n^2-4n+7} + n - 3\right)\right]^1\right\}$,

DL-spec$(\Gamma_H) = \left\{[0]^1, \left[\frac{n+1}{2}\right]^1, [n]^{\frac{n-3}{2}}\right\}$,

DQ-spec$(\Gamma_H) = \Big\{[n-4]^{\frac{n-3}{2}}, \left[\frac{1}{4}\left(-\sqrt{9n^2-46n+73} + 5n - 11\right)\right]^1,$
$\left[\frac{1}{4}\left(\sqrt{9n^2-46n+73} + 5n - 11\right)\right]^1\Big\}$

and $W(\Gamma_H) = \frac{(n-1)^2}{4}$.

(b). *If n and $\frac{n}{2}$ are even then*

D-spec$(\Gamma_H) = \Big\{[-2]^{\frac{n-4}{2}}, [-1]^1, \left[\frac{1}{2}\left(-\sqrt{n^2-6n+17} + n - 3\right)\right]^1,$
$\left[\frac{1}{2}\left(\sqrt{n^2-6n+17} + n - 3\right)\right]^1\Big\}$,

DL-spec$(\Gamma_H) = \left\{[0]^1, \left[\frac{n+2}{2}\right]^2, [n]^{\frac{n-4}{2}}\right\}$,

DQ-spec$(\Gamma_H) = \Big\{[n-4]^{\frac{n-4}{2}}, \left[\frac{n-2}{2}\right]^1, \left[\frac{1}{4}\left(-\sqrt{9n^2-68n+164} + 5n - 10\right)\right]^1,$
$\left[\frac{1}{4}\left(\sqrt{9n^2-68n+164} + 5n - 10\right)\right]^1\Big\}$

and $W(\Gamma_H) = \frac{(n^2-2n+4)}{4}$.

(c). *If n is even and $\frac{n}{2}$ is odd then*

D-spec$(\Gamma_H) = \Big\{[-2]^{\frac{n-2}{2}}, \left[\frac{1}{2}\left(-\sqrt{n^2-8n+28} + n - 2\right)\right]^1,$
$\left[\frac{1}{2}\left(\sqrt{n^2-8n+28} + n - 2\right)\right]^1\Big\}$,

DL-spec$(\Gamma_H) = \left\{[0]^1, \left[\frac{n+2}{2}\right]^1, [n]^{\frac{n-4}{2}}, \left[\frac{n+6}{2}\right]^1\right\}$,

DQ-spec$(\Gamma_H) = \Big\{[n-4]^{\frac{n-4}{2}}, \left[\frac{n-2}{2}\right]^1, \left[\frac{1}{4}\left(-\sqrt{9n^2-92n+292} + 5n - 6\right)\right]^1,$
$\left[\frac{1}{4}\left(\sqrt{9n^2-92n+292} + 5n - 6\right)\right]^1\Big\}$

and $W(\Gamma_H) = \frac{(n^2-2n+8)}{4}$.

Proof. We know that

$$|Z(H)| = \begin{cases} 1, & \text{for } n \text{ is odd} \\ 2, & \text{for } n \text{ is even} \end{cases}$$

and

$$\frac{H}{Z(H)} \cong \begin{cases} D_{2n}, & \text{for } n \text{ is odd} \\ D_{2\times 2}, & \text{for } n = 4 \\ D_{2\times \frac{n}{2}}, & \text{for } n \text{ is even and } n \geq 6. \end{cases}$$

Now, by using Theorem 2.3 and Theorem 2.5, we get the required result. □

Remark 2.7. *Given any perfect square k^2 (where $k \geq 1$) if we consider the group $H = D_{2(2k+1)}$ then by Corollary 2.6(a) we have $W(\Gamma_H) = \frac{(2k+1-1)^2}{4} = k^2$. This shows that every perfect square can be viewed as Wiener index of NCCC-graphs of some dihedral groups. Hence, Inverse Wiener index Problem is solved for NCCC-graphs of finite groups when n is a perfect square. However, it may be challenging to solve Inverse Wiener index Problem in general for NCCC-graphs of finite groups.*

Corollary 2.8. *Let H be the group $U_{(n,m)}$, where $m \geq 3$ and $n \geq 2$.*

(a). *If m is odd then*

$$\text{D-spec}(\Gamma_H) = \Big\{[-2]^{\frac{1}{2}(mn+n-4)}, \left[\tfrac{1}{2}\left(-\sqrt{m^2-4m+7}n+mn+n-4\right)\right]^1, \left[\tfrac{1}{2}\left(\sqrt{m^2-4m+7}n+mn+n-4\right)\right]^1\Big\},$$

$$\text{DL-spec}(\Gamma_H) = \Big\{[0]^1, \left[\tfrac{1}{2}(m+1)n\right]^1, [mn]^{\frac{1}{2}(m-1)n-1}, \left[\tfrac{1}{2}(m+3)n\right]^{n-1}\Big\},$$

$$\text{DQ-spec}(\Gamma_H) = \Big\{[mn-4]^{\frac{1}{2}(m-1)n-1}, \left[\tfrac{1}{2}(m+3)n-4\right]^{n-1}, \left[\tfrac{1}{4}\left(-\sqrt{9m^2-46m+73}n+5mn+5n-16\right)\right]^1, \left[\tfrac{1}{4}\left(\sqrt{9m^2-46m+73}n+5mn+5n-16\right)\right]^1\Big\}$$

and $W(\Gamma_H) = \frac{1}{4}n\left(m^2n-2m+3n-2\right)$.

(b). *If m and $\frac{m}{2}$ are even then*

$$\text{D-spec}(\Gamma_H)=\Big\{[-2]^{\frac{mn}{2}+n-3}, [n-2]^1, \left[\tfrac{1}{4}\left(-2\sqrt{m^2-6m+17}n+2mn+2n-8\right)\right]^1, \left[\tfrac{1}{4}\left(2\sqrt{m^2-6m+17}n+2mn+2n-8\right)\right]^1\Big\},$$

$$\text{DL-spec}(\Gamma_H) = \Big\{[0]^1, \left[\tfrac{1}{2}(m+2)n\right]^2, [mn]^{\frac{1}{2}(m-2)n-1}, \left[\tfrac{1}{2}(m+4)n\right]^{2(n-1)}\Big\},$$

$$\text{DQ-spec}(\Gamma_H) = \Big\{[mn-4]^{\frac{1}{2}(m-2)n-1}, \left[\tfrac{1}{2}(m+4)n-4\right]^{2n-2}, \left[\tfrac{1}{2}(m+6)n-4\right]^1, \left[-\tfrac{1}{4}\left(\sqrt{9m^2-68m+164}-5m-6\right)n-4\right]^1, \left[\tfrac{1}{4}\left(\sqrt{9m^2-68m+164}+5m+6\right)n-4\right]^1\Big\}$$

and $W(\Gamma_H) = \frac{1}{4}n\left(m^2n-2m+8n-4\right)$.

(c). *If m is even and $\frac{m}{2}$ is odd then*

$$\text{D-spec}(\Gamma_H) = \Big\{[-2]^{\frac{mn}{2}+n-2}, \left[-\tfrac{1}{2}\left(\sqrt{m^2-8m+28}-m-2\right)n-2\right]^1, \left[\tfrac{1}{2}\left(\sqrt{m^2-8m+28}+m+2\right)n-2\right]^1\Big\},$$

$$\text{DL-spec}(\Gamma_H) = \Big\{[0]^1, \left[\tfrac{1}{2}(m+2)n\right]^1, [mn]^{\frac{1}{2}(m-2)n-1}, \left[\tfrac{1}{2}(m+6)n\right]^{2n-1}\Big\},$$

$$\text{DQ-spec}(\Gamma_H) = \Big\{[mn-4]^{\frac{1}{2}(m-2)n-1}, \left[\tfrac{1}{2}(m+6)n-4\right]^{2n-1}, \left[-\tfrac{1}{4}\left(\sqrt{9m^2-92m+292}-5m-10\right)n-4\right]^1, \left[\tfrac{1}{4}\left(\sqrt{9m^2-92m+292}+5m+10\right)n-4\right]^1\Big\}$$

and $W(\Gamma_H) = \frac{1}{4}n\left(m^2n-2m+12n-4\right)$.

Proof. We know that

$$|Z(H)| = \begin{cases} n, & \text{for } m \text{ is odd} \\ 2n, & \text{for } m \text{ is even} \end{cases}$$

and

$$\frac{H}{Z(H)} \cong \begin{cases} D_{2m}, & \text{for } m \text{ is odd} \\ D_{2\times 2}, & \text{for } m = 4 \\ D_{2\times \frac{m}{2}}, & \text{for } m \text{ is even and } m \geq 8. \end{cases}$$

Hence, by using Theorem 2.3 and Theorem 2.5, we get the required result. □

Remark 2.9. *If $H = U_{(n,m)}$ then, as mentioned above, $\frac{H}{Z(H)}$ is isomorphic to D_{2m} or $D_{2\times\frac{m}{2}}$ according as m is odd or m is even. Therefore, by [36, Theorem 1.2], we have*

$$\Gamma_H = \begin{cases} K_{\frac{(m-1)n}{2}, n}, & \textit{if } m \textit{ is odd} \\ K_{\frac{(m-2)n}{2}, n, n}, & \textit{if } m \textit{ and } \frac{m}{2} \textit{ are even} \\ K_{\frac{(m-2)n}{2}, 2n}, & \textit{if if } m \textit{ is even and } \frac{m}{2} \textit{ is odd.} \end{cases}$$

It is worth mentioning that the cases when $\frac{m}{2}$ is even or odd were not considered in [38, Proposition 2.3]. Therefore, the structure of $\Gamma_{U_{(n,m)}}$ described in [38, Proposition 2.3] is not complete.

Corollary 2.10. *Let H be the group T_{4n}, where $n \geq 2$.*

(a). *If n is even then*

$$\text{D-spec}(\Gamma_H) = \{[-2]^{n-2}, [-1]^1, [\tfrac{1}{4}(-2\sqrt{4n^2-12n+17}+4n-6)]^1, [\tfrac{1}{4}(2\sqrt{4n^2-12n+17}+4n-6)]^1\},$$

$$\text{DL-spec}(\Gamma_H) = \{[0]^1, [n+1]^2, [2n]^{n-2}\},$$

$$\text{DQ-spec}(\Gamma_H) = \left\{[2n-4]^{n-2}, [n-1]^1, \left[\tfrac{1}{4}\left(-2\sqrt{9n^2-34n+41}+10n-10\right)\right]^1, \left[\tfrac{1}{4}\left(2\sqrt{9n^2-34n+41}+10n-10\right)\right]^1\right\}$$

and $W(\Gamma_H) = n^2 - n + 1$.

(b). *If n is odd then*

$$\text{D-spec}(\Gamma_H) = \left\{[-2]^{n-1}, \left[\tfrac{1}{2}\left(-2\sqrt{n^2-4n+7}+2n-2\right)\right]^1, \left[\tfrac{1}{2}\left(2\sqrt{n^2-4n+7}+2n-2\right)\right]^1\right\},$$

$$\text{DL-spec}(\Gamma_H) = \{[0]^1, [n+1]^1, [2n]^{n-2}, [n+3]^1\},$$

$$\text{DQ-spec}(\Gamma_H) = \left\{[2(n-2)]^{n-2}, [n-1]^1, \left[\tfrac{1}{4}\left(-2\sqrt{9n^2-46n+73}+10n-6\right)\right]^1, \left[\tfrac{1}{4}\left(2\sqrt{9n^2-46n+73}+10n-6\right)\right]^1\right\}$$

and $W(\Gamma_H) = n^2 - n + 2$.

Proof. We know that $|Z(H)| = 2$ and $\frac{H}{Z(H)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or D_{2n} according as $n = 2$ or $n \geq 3$. Hence, by using Theorem 2.3 and Theorem 2.5, we get the required result. □

Corollary 2.11. *Let H be the semidihedral group SD_{8n}, where $n \geq 2$.*

(a). *If n is even then*

$$\text{D-spec}(\Gamma_H) = \{[-2]^{2n-2}, [-1]^1, [\tfrac{1}{4}(-2\sqrt{16n^2-24n+17}+8n-6)]^1, [\tfrac{1}{4}(2\sqrt{16n^2-24n+17}+8n-6)]^1\},$$

$\text{DL-spec}(\Gamma_H) = \{[0]^1, [2n+1]^2, [4n]^{2n-2}\}$,

$\text{DQ-spec}(\Gamma_H) = \Big\{[4n-4]^{2n-2}, [2n-1]^1, \left[\frac{1}{4}\left(-2\sqrt{36n^2-68n+41}+20n-10\right)\right]^1, \left[\frac{1}{4}\left(2\sqrt{36n^2-68n+41}+20n-10\right)\right]^1\Big\}$

and $W(\Gamma_H) = 4n^2 - 2n + 1$.

(b). *If n is odd then*

$\text{D-spec}(\Gamma_H) = \Big\{[-2]^{2n}, \left[\frac{1}{2}\left(4n-4\sqrt{n^2-4n+7}\right)\right]^1, \left[\frac{1}{2}\left(4\sqrt{n^2-4n+7}+4n\right)\right]^1\Big\}$,

$\text{DL-spec}(\Gamma_H) = \{[0]^1, [2(n+1)]^1, [4n]^{2n-3}, [2(n+3)]^3\}$,

$\text{DQ-spec}(\Gamma_H) = \Big\{[4(n-1)]^{2n-3}, [2(n+1)]^3, \left[\frac{1}{4}\left(-4\sqrt{9n^2-46n+73}+20n+4\right)\right]^1, \left[\frac{1}{4}\left(4\sqrt{9n^2-46n+73}+20n+4\right)\right]^1\Big\}$

and $W(\Gamma_H) = 4n^2 - 2n + 10$.

Proof. We know that

$$|Z(H)| = \begin{cases} 2, & \text{for } n \text{ is even} \\ 4, & \text{for } n \text{ is odd} \end{cases}$$

and

$$\frac{H}{Z(H)} \cong \begin{cases} D_{4n}, & \text{for } n \text{ is even} \\ D_{2n}, & \text{for } n \text{ is odd.} \end{cases}$$

Hence, by using Theorem 2.5, we get required result. □

Corollary 2.12. *Let H be the group U_{6n}, where $n \geq 2$. Then*

$\text{D-spec}(\Gamma_H) = \Big\{[-2]^{2n-2}, [n-2]^1, [3n-2]^1\Big\}$, $\text{DL-spec}(\Gamma_H) = \{[0]^1, [2n]^1, [3n]^{2(n-1)}\}$,

$\text{DQ-spec}(\Gamma_H) = \{[3n-4]^{2(n-1)}, [4(n-1)]^1, [6n-4]^1\}$ *and* $W(\Gamma_H) = n(3n-2)$.

Proof. We know that $|Z(H)| = n$ and $\frac{H}{Z(H)} \cong D_{2\times 3}$. Hence, by using Theorem 2.5, we get the required result. □

We conclude this section with the following result.

Theorem 2.13. *Let H be the group V_{8n}, where $n \geq 2$.*

(a). *If n is even then*

$\text{D-spec}(\Gamma_H) = \{[0]^1, [-2]^{2n-1}, [-\sqrt{4n^2-12n+17}+2n-1]^1, [\sqrt{4n^2-12n+17}+2n-1]^1\}$,

$\text{DL-spec}(\Gamma_H) = \{[0]^1, [2n+2]^2, [4n]^{2n-3}, [2n+4]^2\}$,

$\text{DQ-spec}(\Gamma_H) =$

$\Big\{[4n-4]^{2n-3}, [2n]^2, [2n+2]^1, \left[-\sqrt{9n^2-34n+41}+5n-1\right]^1, \left[\sqrt{9n^2-34n+41}+5n-1\right]^1\Big\}$

and $W(\Gamma_H) = 4n^2 - 2n + 6$.

(b). *If n is odd then*

$\text{D-spec}(\Gamma_H) = \Big\{[-1]^1, [-2]^{2n-2}, \left[\frac{1}{2}\left(-\sqrt{16n^2-24n+17}+4n-3\right)\right]^1, \left[\frac{1}{2}\left(\sqrt{16n^2-24n+17}+4n-3\right)\right]^1\Big\}$,

$$\text{DL-spec}(\Gamma_H) = \left\{[0]^1, [4n]^{2n-2}, [2n+1]^2\right\},$$
$$\text{DQ-spec}(\Gamma_H) = \Big\{[4n-4]^{2n-2}, [2n-1]^1, \left[\tfrac{1}{2}\left(-\sqrt{36n^2-68n+41}+10n-5\right)\right]^1, \left[\tfrac{1}{2}\left(\sqrt{36n^2-68n+41}+10n-5\right)\right]^1\Big\}$$

and $W(\Gamma_H) = 4n^2 - 2n + 1$.

Proof. (a) If n is even then, by [38, Proposition 2.4], we have $\Gamma_H = K_{2n-2,2,2}$. Here, $|V(\Gamma_H)| = 2(n+1)$. Using Lemma 2.2(a), we get

$$\begin{aligned}\text{D-spec}(x) &= (x+2)^{2n-1}\left[\prod_{i=1}^{3}(x-n_i+2) - \sum_{i=1}^{3} n_i \prod_{j=1, j\neq i}^{3}(x-n_j+2)\right]\\ &= (x+2)^{2n-1}x\big[x(x-2n+4)-(2n-2)x-4(x-2n+4)\big].\end{aligned}$$

Therefore, $\text{D-spec}(\Gamma_H) = \left\{[0]^1, [-2]^{2n-1}, \left[2n-1-\sqrt{4n^2-12n+17}\right]^1, \left[2n-1+\sqrt{4n^2-12n+17}\right]^1\right\}$.

Using Lemma 2.2(b), we get

$$\begin{aligned}\text{Ch}_{DL}(x) &= x(x-(2n+2))^{3-1}\prod_{i=1}^{3}(x-(2n+2)-n_i)^{n_i-1}\\ &= x(x-2n-2)^2(x-4n)^{2n-3}(x-2n-4)^2.\end{aligned}$$

Therefore, $\text{DL-spec}(\Gamma_H) = \left\{[0]^1, [2n+2]^2, [4n]^{2n-3}, [2n+4]^2\right\}$.

Using Lemma 2.2(c), we also get

$$\begin{aligned}\text{Ch}_{DQ}(x) &= \prod_{i=1}^{3}(x-(2n+2)-n_i+4)^{n_i-1}\left(\prod_{i=1}^{3}(x-(2n+2)-2n_i+4)\right.\\ &\qquad\left. - \sum_{i=1}^{3} n_i \prod_{j=1, j\neq i}^{3}(x-(2n+2)-2n_j+4)\right)\\ &= (x-4n+4)^{2n-3}(x-2n)^2(x-2n-2)\left((x-6n+6)(x-2n-2)\right.\\ &\qquad\left. -(2n-2)(x-2n-2)-4(x-6n+6)\right).\end{aligned}$$

Thus $\text{DQ-spec}(\Gamma_H) = \left\{[4n-4]^{2n-3}, [2n]^2, [2n+2]^1, [5n-1-\sqrt{9n^2-34n+41}]^1, [5n-1+\sqrt{9n^2-34n+41}]^1\right\}$.

(b). If n is odd, then by [38, Proposition 2.4], we have $\Gamma_H = K_{2n-1,1,1} = \Gamma_{D_{2\times 4n}}$. Hence, the result follows from Corollary 2.6. □

In the rest part of this section, we characterize various groups considered above such that their NCCC-graph is D-integral, DL-integral and DQ-integral. By Theorem 2.3 and Corollary 2.12, the following result follows immediately.

Theorem 2.14. *Let H be a finite non-abelian group. Then Γ_H is D-integral, DL-integral and DQ-integral if*

(a). $\frac{H}{Z(H)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, *where p is any prime.*

(b). *H is isomorphic to U_{6n}.*

In view of the expressions for DL-spec(Γ_H), in Corollary 2.6 – Corollary 2.12 and Theorem 2.13, the following result follows.

Theorem 2.15. *Let $H = D_{2n}$ (where $n \geq 3$), $U_{(n,m)}$ (where $m \geq 3$ and $n \geq 2$), T_{4n} (where $n \geq 2$), SD_{8n} (where $n \geq 2$) and V_{8n} (where $n \geq 2$). Then Γ_H is DL-integral.*

The following lemma is useful in characterizing the groups considered in Theorem 2.15 such that their NCCC-graphs are D-integral and DQ-integral.

Lemma 2.16. *Let n be any positive integer. Then*

(a). *$n^2 - 4n + 7$ is perfect square if and only if $n = 1, 3$.*

(b). *$n^2 - 8n + 28$ is perfect square if and only if $n = 2, 6$.*

(c). *$4n^2 - 12n + 17$ is perfect square if and only if $n = 1, 2$.*

(d). *$9n^2 - 46n + 73$ is perfect square if and only if $n = 1, 3, 6$.*

(e). *$9n^2 - 34n + 41$ is perfect square if and only if $n = 1, 2, 4$.*

(f). *$9n^2 - 68n + 164$ is perfect square if and only if $n = 2, 4, 5, 8$.*

(g). *$9n^2 - 92n + 292$ is perfect square if and only if $n = 2, 6, 12$.*

Proof. (a). Let $n^2 - 4n + 7$ be a perfect square. Then there exist integers k such that $n^2 - 4n + 7 = k^2$ which gives $(k + n - 2)(k - n + 2) = 3$. Therefore, we have the following cases.

Case 1. $k + n - 2 = 1$ and $k - n + 2 = 3$

In this case, we have $k + n = 3$ and $k - n = 1$ which gives $k = 2$ and $n = 1$.

Case 2. $k + n - 2 = -1$ and $k - n + 2 = -3$

In this case, we have $k + n = 1$ and $k - n = -5$ which gives $k = -2$ and $n = 3$.

Case 3. $k + n - 2 = 3$ and $k - n + 2 = 1$

In this case, we have $k + n = 5$ and $k - n = -1$ which gives $k = 2$ and $n = 3$.

Case 4. $k + n - 2 = -3$ and $k - n + 2 = -1$

In this case, we have $k + n = -1$ and $k - n = -3$ which gives $k = -2$ and $n = 1$.

Hence, the result follows.

(b). If $n^2 - 8n + 28$ is a perfect square then there exist integers k such that $n^2 - 8n + 28 = k^2$ which gives $(k + n - 4)(k - n + 4) = 12$. By considering various cases as above we get $n = 2, 6$. Hence, the result follows.

(c) If $4n^2 - 12n + 17$ is a perfect square then there exist integers k such that $4n^2 - 12n + 17 = k^2$ which gives $(k + 2n - 3)(k - 2n + 3) = 8$. By considering various cases as above we get $n = 1, 2$. Hence, the result follows.

(d) Let $9n^2 - 46n + 73$ be a perfect square. Then there exist integers k such that $9n^2 - 46n + 73 = k^2$ which implies $9n^2 - 46n + (73 - k^2) = 0$. Since n is a positive integer the discriminant $\Omega = (-46)^2 - 4 \times 9 \times (73 - k^2) = 36k^2 - 512$ of the quadratic equation must be a perfect square. Let $36k^2 - 512 = a^2$ for some integers a. Then we get $(6k + a)(6k - a) = 512$. If $(6k + a) = 1$ then $(6k - a) = 512$ and so $k = \frac{513}{12}$ and $a = \frac{-511}{2}$; a contradiction. If $6k + a = 2$ then $6k - a = 256$ and so $k = \frac{258}{12}$ and $a = -127$; a contradiction. Similarly, it can be seen that the cases when $(6k + a) = -1$, $(6k - a) = -512$ and $6k + a = -2$, $6k - a = 256$ are not possible. Therefore, without loss of generality, we consider the following cases.

Case 1. $6k + a = 4$ and $6k - a = 128$. In this case, we get $k = 11$ and $a = -62$.

Case 2. $6k + a = -4$ and $6k - a = -128$. In this case, we get $k = -11$ and $a = 62$.

Case 3. $6k + a = 8$ and $6k - a = 64$. In this case, we get $k = 6$ and $a = -28$.

Case 4. $6k + a = -8$ and $6k - a = -64$. In this case, we get $k = -6$ and $a = 28$.

Case 5. $6k + a = 16$ and $6k - a = 32$. In this case, we get $k = 4$ and $a = -8$.

Case 6. $6k + a = -16$ and $6k - a = -32$. In this case, we get $k = -4$ and $a = 8$.

Thus the possible values of k are $\pm 4, \pm 6$ and ± 11. Therefore, $9n^2 - 46n + 73 = 16$, $9n^2 - 46n + 73 = 36$ and $9n^2 - 46n + 73 = 121$. On solving these equations we get $n = 1, 3, 6$. Therefore, $9n^2 - 46n + 73$ is perfect square if and only if $n = 1, 3, 6$.

(e) If $9n^2 - 34n + 41$ is a perfect square then there exist integers k such that $9n^2 - 34n + 41 = k^2$. The discriminant of this quadratic equation is $\Omega = 36k^2 - 320$. Let $36k^2 - 320 = a^2$ for some integer a. Then $(6k + a)(6k - a) = 320$. Now by considering various cases as in the proof of part (d) we get $k = \pm 3, \pm 4, \pm 7$. Therefore, $9n^2 - 34n + 41 = 9$, $9n^2 - 34n + 41 = 16$ and $9n^2 - 34n + 41 = 49$. On solving these equations we get $n = 1, 2, 4$. Therefore, $9n^2 - 34n + 41$ is a perfect square if and only if $n = 1, 2, 4$.

(f) If $9n^2 - 68n + 164$ is a perfect square then there exist integers k such that $9n^2 - 68n + 164 = k^2$. The discriminant of this quadratic equation is $\Omega = 36k^2 - 1280$. Let $36k^2 - 1280 = a^2$ for some integer a. Then $(6k+a)(6k-a) = 1280$. Now by considering various cases as above we get $k = \pm 6, \pm 7, \pm 8, \pm 14, \pm 27$. Therefore, $9n^2 - 68n + 164 = 36$, $9n^2 - 68n + 164 = 49$, $9n^2 - 68n + 164 = 64$, $9n^2 - 68n + 164 = 196$ and $9n^2 - 68n + 164 = 729$. On solving these equations we get $n = 2, 4, 5, 8$. Therefore, $9n^2 - 68n + 164$ is a perfect square if and only if $n = 2, 4, 5, 8$.

(g) If $9n^2 - 92n + 292$ is a perfect square then there exist integers k such that $9n^2 - 92n + 292 = k^2$. The discriminant of this quadratic equation is $\Omega = 36k^2 - 2048$. Let $36k^2 - 2048 = a^2$ for some integer a. Then $(6k + a)(6k - a) = 2048$. Now by considering various cases as above we get $k = \pm 8, \pm 12, \pm 22, \pm 43$. Therefore, $9n^2 - 92n + 292 = 64$, $9n^2 - 92n + 292 = 144$, $9n^2 - 92n + 292 = 484$ and $9n^2 - 92n + 292 = 1849$. On solving these equations we get $n = 2, 6, 12$. Therefore, $9n^2 - 68n + 164$ is perfect square if and only if $n = 2, 6, 12$. □

We conclude this section with the following characterization.

Theorem 2.17. *Let $H = D_{2n}$ (where $n \geq 3$), $U_{(n,m)}$ (where $m \geq 3$ and $n \geq 2$), T_{4n} (where $n \geq 2$), SD_{8n} (where $n \geq 2$) and V_{8n} (where $n \geq 2$). Then*

(a). *Γ_H is D-integral if and only if $H = D_6$, D_8, D_{12}, $U_{(n,3)}$, $U_{(n,4)}$, $U_{(n,6)}$, T_8, T_{12}, SD_{24}, V_{16} and U_{6n} for $n \geq 2$.*

(b). *Γ_H is DQ-integral if and only if $H = D_6$, D_8, D_{12}, D_{16}, $U_{(n,3)}$, $U_{(n,4)}$, $U_{(n,6)}$, $U_{(n,8)}$, T_8, T_{12}, T_{16}, SD_{16}, SD_{24}, V_{16}, V_{32} and U_{6n} for $n \geq 2$.*

Proof. (a). Consider the following cases.

Case 1. $H = D_{2n}$, where $n \geq 3$.

If n is odd then, by Corollary 2.6(a), it is sufficient to show that $\frac{1}{2}(-\sqrt{n^2 - 4n + 7} + n - 3)$ and $\frac{1}{2}(\sqrt{n^2 - 4n + 7} + n - 3)$ are integers. By Lemma 2.16(a), we have $n = 3$ and so $\frac{1}{2}(-\sqrt{n^2 - 4n + 7} + n - 3) = -1$ and $\frac{1}{2}(\sqrt{n^2 - 4n + 7} + n - 3) = 1$. Therefore, if n is odd then Γ_H is D-integral if and only if $H = D_6$.

If n and $\frac{n}{2}$ are even then, in view of Corollary 2.6(b), it is sufficient to show that $\frac{1}{2}(-\sqrt{n^2 - 6n + 17} + n - 3)$ and $\frac{1}{2}\left(\sqrt{n^2 - 6n + 17} + n - 3\right)$ are integers. Putting $n = \frac{n}{2}$ in Lemma 2.16(c) we get that $n^2 - 6n + 17$ is a perfect square if and only if $n = 4$. Therefore, $\frac{1}{2}(-\sqrt{n^2 - 6n + 17} + n - 3) = -1$ and $\frac{1}{2}\left(\sqrt{n^2 - 6n + 17} + n - 3\right) = 2$. So, in this case Γ_H is D-integral if and only if $H = D_8$.

If n is even and $\frac{n}{2}$ is odd then, in view of Corollary 2.6(c), it is sufficient to show that $\frac{1}{2}(n-2-\sqrt{n^2-8n+28})$ and $\frac{1}{2}(\sqrt{n^2-8n+28}+n-2)$ are integers. By Lemma 2.16(b) we have $n=6$ and so $\frac{1}{2}(-\sqrt{n^2-8n+28}+n-2)=0$ and $\frac{1}{2}(\sqrt{n^2-8n+28}+n-2)=4$. Therefore, If n is even and $\frac{n}{2}$ is odd then Γ_H is D-integral if and only if $H=D_{12}$.

Case 2. $H=U_{(n,m)}$, where $m\geq 3$ and $n\geq 2$.

If m is odd then, by Corollary 2.8(a), it is sufficient to show that $\frac{1}{2}(mn+n-4-\sqrt{m^2-4m+7}n)$ and $\frac{1}{2}(mn+n-4+\sqrt{m^2-4m+7}n)$ are integers. By Lemma 2.16(a), we have $m=3$ and so $\frac{1}{2}\left(-\sqrt{m^2-4m+7}n+mn+n-4\right)=n-2$ and $\frac{1}{2}\left(\sqrt{m^2-4m+7}n+mn+n-4\right)=3n-2$. Therefore, if m is odd then Γ_H is D-integral if and only if $H=U_{(n,3)}$.

If m and $\frac{m}{2}$ are even then, in view of Corollary 2.8(b), it is sufficient to show that $\frac{1}{4}\left(2mn+2n-8-2\sqrt{m^2-6m+17}n\right)$ and $\frac{1}{4}\left(2\sqrt{m^2-6m+17}n+2mn+2n-8\right)$ are integers. Putting $n=\frac{m}{2}$ in Lemma 2.16(c) we get that $m^2-6m+17$ is a perfect square if and only if $m=4$. Therefore, $\frac{1}{4}\left(-2\sqrt{m^2-6m+17}n+2mn+2n-8\right)=n-2$ and $\frac{1}{4}\left(2\sqrt{m^2-6m+17}n+2mn+2n-8\right)=4n-2$. So, in this case Γ_H is D-integral if and only if $H=U_{(n,4)}$.

If m is even and $\frac{m}{2}$ is odd then, in view of Corollary 2.8(c), it is sufficient to show that $-\frac{1}{2}\left(\sqrt{m^2-8m+28}-m-2\right)n-2$ and $\frac{1}{2}\left(\sqrt{m^2-8m+28}+m+2\right)n-2$ are integers. By Lemma 2.16(b) we have $m=6$ and so $-\frac{1}{2}\left(\sqrt{m^2-8m+28}-m-2\right)n-2=2n-2$ and $\frac{1}{2}\left(\sqrt{m^2-8m+28}+m+2\right)n-2=6n-2$. Therefore, If m is even and $\frac{m}{2}$ is odd then Γ_H is D-integral if and only if $H=U_{(n,6)}$.

Case 3. $H=T_{4n}$, where $n\geq 2$.

If n is even then, by Corollary 2.10(a), it is sufficient to show that $\frac{1}{4}(-2\sqrt{4n^2-12n+17}+4n-6)$ and $\frac{1}{4}(2\sqrt{4n^2-12n+17}+4n-6)$ are integers. By Lemma 2.16(c), we have $n=2$ and so $\frac{1}{4}(-2\sqrt{4n^2-12n+17}+4n-6)=-1$ and $\frac{1}{4}(2\sqrt{4n^2-12n+17}+4n-6)=2$. Therefore, if n is even then Γ_H is D-integral if and only if $H=T_8$.

If n is odd then, in view of Corollary 2.10(b), it is sufficient to show that $\frac{1}{2}(2n-2-2\sqrt{n^2-4n+7})$ and $\frac{1}{2}(2n-2+2\sqrt{n^2-4n+7})$ are integers. By Lemma 2.16(a), we have $n=3$ and so $\frac{1}{2}\left(2n-2-2\sqrt{n^2-4n+7}\right)=0$ and $\frac{1}{2}\left(2\sqrt{n^2-4n+7}+2n-2\right)=4$. So, in this case Γ_H is D-integral if and only if $H=T_{12}$.

Case 4. $H=SD_{8n}$, where $n\geq 2$.

If n is even then, by Corollary 2.11(a), it is sufficient to show that $\frac{1}{4}(8n-6-2\sqrt{16n^2-24n+17})$ and $\frac{1}{4}(8n-6+2\sqrt{16n^2-24n+17})$ are integers. Putting $n=2n$ in Lemma 2.16(c), we get that $16n^2-24n+17$ is a perfect square if and only if $n=1$. Therefore, if n is even then Γ_H is not D-integral.

If n is odd then, in view of Corollary 2.11(b), it is sufficient to show that $\frac{1}{2}(4n-4\sqrt{n^2-4n+7})$ and $\frac{1}{2}(4\sqrt{n^2-4n+7}+4n)$ are integers. By Lemma 2.16(a), we have $n=3$ and so $\frac{1}{2}(4n-4\sqrt{n^2-4n+7})=2$ and $\frac{1}{2}(4\sqrt{n^2-4n+7}+4n)=10$. So, in this case Γ_H is D-integral if and only if $H=SD_{24}$.

Case 5. $H=V_{8n}$, where $n\geq 2$.

If n is even then, in view of Corollary 2.13(a), it is sufficient to show that $2n-1-\sqrt{4n^2-12n+17}$ and $2n-1+\sqrt{4n^2-12n+17}$ are integers. By Lemma 2.16(c), we have $n=2$ and so $-\sqrt{4n^2-12n+17}+2n-1=0$ and $\sqrt{4n^2-12n+17}+2n-1=6$. So, in this case Γ_H is D-integral if and only if $H=V_{16}$.

If n is odd then, by Corollary 2.13(b), it is sufficient to show that $\frac{1}{2}(4n-3-\sqrt{16n^2-24n+17})$ and $\frac{1}{2}(4n-3+\sqrt{16n^2-24n+17})$ are integers. Putting $n=2n$ in Lemma 2.16(c), we get that $16n^2-24n+17$ is a perfect square if and only if $n=1$. Therefore, if n is odd then Γ_H is not D-integral.

Case 6. $H=U_{6n}$, where $n\geq 2$.

By Corollary 2.12, it follows that Γ_H is D-integral for $n\geq 2$.

(b). Consider the following cases.

Case 1. $H = D_{2n}$, where $n \geq 3$.

If n is odd then, by Corollary 2.6(a), it is sufficient to show that $\frac{1}{4}(-9\sqrt{9n^2-46n+73}+5n-11)$ and $\frac{1}{4}(9\sqrt{9n^2-46n+73}+5n-11)$ are integers. By Lemma 2.16(d), we have $n=3$ and so $\frac{1}{4}(-9\sqrt{9n^2-46n+73}+5n-11)=0$ and $\frac{1}{4}(9\sqrt{9n^2-46n+73}+5n-11)=0$. If n is odd then Γ_H is DQ-integral if and only if $H = D_6$.

Note that $\frac{n-2}{2}$ is an integer if n is even. If n and $\frac{n}{2}$ are even then, in view of Corollary 2.6(b), it is sufficient to show that $\frac{1}{4}(-\sqrt{9n^2-68n+164}+5n-10)$ and $\frac{1}{4}(\sqrt{9n^2-68n+164}+5n-10)$ are integers. By Lemma 2.16(f), we have $n=4,8$ and so $\frac{1}{4}(-\sqrt{9n^2-68n+164}+5n-10)=1$ or 4 and $\frac{1}{4}(\sqrt{9n^2-68n+164}+5n-10)=4$ or 11 according as $n=4$ or $n=8$. Therefore, if n and $\frac{n}{2}$ are even then, Γ_H is DQ-integral if and only if $H = D_8, D_{16}$.

If n is even and $\frac{n}{2}$ is odd then, in view of Corollary 2.6(c), it is sufficient to show that $\frac{1}{4}(5n-6-\sqrt{9n^2-92n+292})$ and $\frac{1}{4}(\sqrt{9n^2-92n+292}+5n-6)$ are integers. By Lemma 2.16(g), we have $n=6$ and so $\frac{1}{4}(-\sqrt{9n^2-92n+292}+5n-6)=4$ and $\frac{1}{4}(\sqrt{9n^2-92n+292}+5n-6)=8$. Therefore, Γ_H is DQ-integral if and only if $H = D_{12}$.

Case 2. $H = U_{(n,m)}$, where $m \geq 3$ and $n \geq 2$.

If m is odd then, by Corollary 2.8(a), it is sufficient to show that $\frac{1}{4}(5mn+5n-16-\sqrt{9m^2-46m+73}n)$ and $\frac{1}{4}(5mn+5n-16+\sqrt{9m^2-46m+73}n)$ are integers. By Lemma 2.16(d), we have $m=3$ and so $\frac{1}{4}(5mn+5n-16-\sqrt{9m^2-46m+73}n)=4(n-1)$ and $\frac{1}{4}(5mn+5n-16+\sqrt{9m^2-46m+73}n)=6n-4$. Again for m is odd, $\frac{(m+3)n}{2}-4$ is integer. So if m is odd then Γ_H is DQ-integral if and only if $H = U_{(n,3)}$.

Note that $\frac{(m+4)n}{2}-4$ and $\frac{(m+6)n}{2}-4$ are integers if m is even. If m and $\frac{m}{2}$ are even then, in view of Corollary 2.8(b), it is sufficient to show that $-\frac{1}{4}(\sqrt{9m^2-68m+164}-5m-6)n-4$ and $\frac{1}{4}\left(\sqrt{9m^2-68m+164}+5m+6\right)n-4$ are integers. By Lemma 2.16(f), we have $m=4,8$ and so $-\frac{1}{4}(\sqrt{9m^2-68m+164}-5m-6)n-4=5n-4$ or $8n-4$ and $\frac{1}{4}\left(\sqrt{9m^2-68m+164}+5m+6\right)n-4=8n-4$ or $15n-4$ according as $m=4$ or $m=8$. Therefore, if m and $\frac{m}{2}$ are even then, Γ_H is DQ-integral if and only if $H = U_{(n,4)}, U_{(n,8)}$.

If m is even and $\frac{m}{2}$ is odd then, in view of Corollary 2.8(c), it is sufficient to show that $-\frac{1}{4}\left(\sqrt{9m^2-92m+292}-5m-10\right)n-4$ and $\frac{1}{4}\left(\sqrt{9m^2-92m+292}+5m+10\right)n-4$ are integers. By Lemma 2.16(g), we have $m=6$ and so $-\frac{1}{4}(\sqrt{9m^2-92m+292}-5m-10)n-4=8n-4$ and $\frac{1}{4}(5m+10+\sqrt{9m^2-92m+292})n-4=12n-4$. Again for m is even $\frac{(m+6)n}{2}-4$ is an integer. Therefore, Γ_H is DQ-integral if and only if $H = U_{(n,6)}$.

Case 3. $H = T_{4n}$, where $n \geq 2$.

If n is even then, by Corollary 2.10(a), it is sufficient to show that $\frac{1}{4}(-2\sqrt{9n^2-34n+41}+10n-10)$ and $\frac{1}{4}(2\sqrt{9n^2-34n+41}+10n-10)$ are integers. By Lemma 2.16(e), we have $n=2,4$ and so $\frac{1}{4}(-2\sqrt{9n^2-34n+41}+10n-10)=1$ or 4 and $\frac{1}{4}(2\sqrt{9n^2-34n+41}+10n-10)=4$ or 11 according as $n=2$ or $n=4$. If n is even then Γ_H is DQ-integral if and only if $H = T_8, T_{16}$.

If n is odd then, by Corollary 2.10(b), it is sufficient to show that $\frac{1}{4}(10n-6-2\sqrt{9n^2-46n+73})$ and $\frac{1}{4}(10n-6+2\sqrt{9n^2-46n+73})$ are integers. By Lemma 2.16(d), we have $n=3$ and so $\frac{1}{4}(-2\sqrt{9n^2-46n+73}+10n-6)=4$ and $\frac{1}{4}(10n-6+2\sqrt{9n^2-46n+73})=8$. If n is odd then Γ_H is DQ-integral if and only if $H = T_{12}$.

Case 4. $H = SD_{8n}$, where $n \geq 2$.

If n is even then, by Corollary 2.11(a), it is sufficient to show that $\frac{1}{4}(20n-10-2\sqrt{36n^2-68n+41})$ and $\frac{1}{4}(2\sqrt{36n^2-68n+41}+20n-10)$ are integers. Putting $n=2n$ in Lemma 2.16(e), we have $n=2$ and so $\frac{1}{4}(-2\sqrt{36n^2-68n+41}+20n-10)=4$ and $\frac{1}{4}(2\sqrt{36n^2-68n+41}+20n-10)=11$. If n is even then

Γ_H is DQ-integral if and only if $H = SD_{16}$.

If n is odd then, by Corollary 2.11(b), it is sufficient to show that $\frac{1}{4}(20n+4-4\sqrt{9n^2-46n+73})$ and $\frac{1}{4}(20n+4+4\sqrt{9n^2-46n+73})$ are integers. By Lemma 2.16(d), we have $n=3$ and so $\frac{1}{4}(20n+4-4\sqrt{9n^2-46n+73}) = 12$ and $\frac{1}{4}(20n+4-4\sqrt{9n^2-46n+73}) = 20$. If n is odd then Γ_H is DQ-integral if and only if $H = SD_{24}$.

Case 5. $H = V_{8n}$, where $n \geq 2$.

If n is even then, by Corollary 2.13(a), it is sufficient to show that $-\sqrt{9n^2-34n+41}+5n-1$ and $\sqrt{9n^2-34n+41}+5n-1$ are integers. By Lemma 2.16(e), we have $n=2,4$ and so $-\sqrt{9n^2-34n+41}+5n-1 = 6$ or 12 and $\sqrt{9n^2-34n+41}+5n-1 = 12$ or 26 according as $n=2$ or $n=4$. If n is even then Γ_H is DQ-integral if and only if $H = V_{16}, V_{32}$.

If n is odd then, by Corollary 2.13(b), it is sufficient to show that $\frac{1}{2}(10n-5-\sqrt{36n^2-68n+41})$ and $\frac{1}{2}(10n-5+\sqrt{36n^2-68n+41})$ are integers. Putting $n=2n$ in Lemma 2.16(e), we get that $36n^2-68n+41$ is a perfect square if and only if $n=1$. Therefore, if n is odd then Γ_H is not DQ-integral.

Case 6. $H = U_{6n}$, where $n \geq 2$.

By Corollary 2.12, it follows that Γ_H is DQ-integral for $n \geq 2$. □

3. Various distance energies

In this section we compute various distance energies of NCCC-graphs of the groups considered in Section 2. In view of (1), we have $\frac{tr(DL(\mathcal{G}))}{|V(\mathcal{G}|} = \frac{2W(\mathcal{G})}{|V(\mathcal{G}|} = \frac{tr(DQ(\mathcal{G}))}{|V(\mathcal{G}|}$. Therefore,

$$E_{DL}(\mathcal{G}) = \sum_{\beta \in \text{DL-spec}(\mathcal{G})} |\beta - \Delta(\mathcal{G})| \text{ and } E_{DQ}(\mathcal{G}) = \sum_{\gamma \in \text{DQ-spec}(\mathcal{G})} |\gamma - \Delta(\mathcal{G})|,$$

where $\Delta(\mathcal{G}) = \frac{2W(\mathcal{G})}{|V(\mathcal{G}|}$. Thus, $W(\mathcal{G})$ plays a crucial role in computing distance Laplacian and signless Laplacian energies of $\mathcal{G}$.

We begin with the class of finite groups whose central quotients are isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for any prime p.

Theorem 3.1. *Let H be a finite non-abelian group such that $\frac{H}{Z(H)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is any prime. If $n = \frac{(p-1)z}{p}$, where $z = |Z(H)|$, then*

$$E_D(\Gamma_H) = E_{DL}(\Gamma_H) = E_{DQ}(\Gamma_H) = \begin{cases} 4, & \text{for } n = 1 \\ 4(n-1)(p+1), & \text{for } n \geq 2. \end{cases}$$

Proof. By Theorem 2.3 we have D-spec$(\Gamma_H) = \{[-2]^{(n-1)(p+1)}, [n-2]^p, [np+2n-2]^1\}$. Therefore,

$$\begin{aligned} E_D(\Gamma_H) &= (n-1)(p+1) \times |-2| + p \times |n-2| + 1 \times |np+2n-2| \\ &= 2(n-1)(p+1) + p \times |n-2| + 1 \times (np+2n-2). \end{aligned}$$

Hence, $E_D(\Gamma_H) = 4$ or $4(n-1)(p+1)$ according as $n=1$ or $n \geq 2$, noting that

$$|n-2| = \begin{cases} 1, & \text{for } n = 1 \\ n-2, & \text{for } n \geq 2. \end{cases}$$

By Theorem 2.3 we have DL-spec$(\Gamma_H) = \{[0]^1, [n(p+1)]^p, [n(p+1)+n]^{(p+1)(n-1)}\}$ and $W(\Gamma_H) =$

$\frac{n(p+1)(n(p+2)-2)}{2}$. Therefore, $\Delta(\Gamma_H) = n(p+2)-2$. We have

$$\begin{aligned}\mathrm{E_{DL}}(\Gamma_H) &= 1\times|0-\Delta(\Gamma_H)| + p\times|n(p+1)-\Delta(\Gamma_H)| + (p+1)(n-1)\times|n(p+1)+n-\Delta(\Gamma_H)|\\ &= |np+2n-2| + p\times|2-n| + (p+1)(n-1)\times|2|.\end{aligned}$$

Hence, $\mathrm{E_{DL}}(\Gamma_H) = 4$ or $4(n-1)(p+1)$ according as $n = 1$ or $n \geq 2$.

By Theorem 2.3 we also have $\text{DQ-spec}(\Gamma_H) = \{[np+2n-4]^{(p+1)(n-1)}, [np+3n-4]^p, [2np+4n-4]^1\}$. We have

$$\begin{aligned}\mathrm{E_{DQ}} = (n-1)(p+1)\times|(np+2n-4)-\Delta(\Gamma_H)| + p\times|(np+3n-4)-\Delta(\Gamma_H)|&\\ +1\times|(2np+4n-4)-\Delta(\Gamma_H)|&\\ = (n-1)(p+1)\times|-2| + p\times|n-2| + |np+2n-2|.&\end{aligned}$$

Therefore, $\text{DQ-spec}(\Gamma_H) = \left\{[np+2n-4]^{(p+1)(n-1)}, [np+3n-4]^p, [2np+4n-4]^1\right\}$. Hence, the result follows. ☐

Corollary 3.2. *Let H be a non-abelian group of order p^n with $|Z(H)| = p^{n-2}$, where p is prime and $n \geq 3$. Then $\mathrm{E_D}(\Gamma_H) = \mathrm{E_{DL}}(\Gamma_H) = \mathrm{E_{DQ}}(\Gamma_H) = \frac{4(p+1)\left(p^{n+1}-p^n-p^3\right)}{p^3}$.*

Remark 3.3. *It is noteworthy that the first couple of equalities in Problem 1.3 were obtained in [13] for NCCC-graphs of groups whose central quotients are isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. We attempt to solve Problem 1.3 by computing various distance energies of NCCC-graphs of this class of groups. Unfortunately, the third equality in Problem 1.3 does not hold though the last couple of equalities hold.*

We now consider the class of finite groups whose central quotients are isomorphic to the dihedral group D_{2n} for $n \geq 3$. This class of groups includes the well-known groups viz. D_{2n} (where $n \geq 3$), $U_{(n,m)}$ (where $m \geq 3$ and $n \geq 2$), T_{4n} (where $n \geq 2$), SD_{8n} (where $n \geq 2$) and U_{6n} (where $n \geq 2$).

Theorem 3.4. *Let H be a finite non-abelian group with $|Z(H)| = z$ and $\frac{H}{Z(H)} \cong D_{2n}$, (where $n \geq 3$).*

(a). *If n is even then*

$$\mathrm{E_D}(\Gamma_H) = \begin{cases} 2n-3+\sqrt{4n^2-12n+17}, & \text{for } z=2\\ 6n-5, & \text{for } z=3\\ 2(nz+z-6), & \text{for } z\geq 4.\end{cases}$$

$$\mathrm{E_{DL}}(\Gamma_H) = \begin{cases} \frac{2\left(2n^2z-2n(z+3)+5z-6\right)}{n+1}, & \text{for } n=4 \ \& \ z=2,3\\ \frac{n^2z^2+2n^2z-3nz^2-2nz-4n+2z^2+2z-4}{n+1}, & \text{otherwise.}\end{cases}$$

$$\mathrm{E_{DQ}}(\Gamma_H) = \begin{cases} \frac{n^2z+n\left(\left(\sqrt{9n^2-34n+41}+8\right)z-8\right)+\left(\sqrt{9n^2-34n+41}-5\right)z-8}{2(n+1)}, & \text{for } z=2 \ \& \ n\geq 4;\\ & z=3 \ \& \ n=4,6;\\ & z=4 \ \& \ n=4\\ \frac{1}{5}\left(6z^2+9z-20\right), & \text{for } n=4 \ \& \ z\geq 5\\ \frac{z\left(n^2(2z-3)+n\left(\sqrt{9n^2-26n+33}-6z+4\right)+\sqrt{9n^2-26n+33}+4z+7\right)}{2(n+1)}, & \text{otherwise.}\end{cases}$$

(b). *If n is odd then*

$$\mathrm{E_D}(\Gamma_H) = \begin{cases} \frac{1}{2}\left(4\sqrt{n^2-4n+7}+n-3\right), & \text{for } z=1\\ \frac{5}{2}(nz+z-4), & \text{for } z\geq 2.\end{cases}$$

$$\mathrm{E_{DL}}(\Gamma_H) = \begin{cases} \frac{2n^2z-4n+6z-4}{n+1}, & \text{for } n=3,5 \ \& \ z=1 \\ \frac{3n^2z-2nz-8n+11z-8}{n+1}, & \text{for } n=7 \ \& \ z=1;\ n=3 \ \& \ z\geq 2; \\ & n=5 \ \& \ z=2 \\ \frac{n^2z^2+2n^2z-4nz^2-2nz-4n+3z^2+4z-4}{n+1}, & \textit{otherwise.} \end{cases}$$

$$\mathrm{E_{DQ}}(\Gamma_H) = \begin{cases} \frac{n^2z+10nz-8n-7z-8}{n+1}, & \text{for } n=3 \ \& \ z\geq 2; \\ & n=5 \ \& \ z=3,4,5 \\ \frac{1}{2}\left(n+\sqrt{n(9n-46)+73}-\frac{16}{n+1}+9\right)z-4, & \text{for } n=3 \ \& \ z=1 \\ \frac{(z-1)(n((n-4)z+4)+3z+4)}{n+1}, & \text{for } n=5 \ \& \ z\geq 6; \\ & n=7 \ \& \ z\geq 56 \\ \frac{1}{2}\left(n+\sqrt{n(9n-46)+73}-\frac{16}{n+1}+9\right)z-4, & \text{for } n=5 \ \& \ z=1,2; \\ & n=7 \ \& \ z=1,2,3 \\ \frac{1}{2}z\left(2\left(\frac{8}{n+1}-5\right)z+n(2z-3)+\sqrt{n(9n-46)+73}+9\right), & \textit{otherwise.} \end{cases}$$

Proof. (a). If n is even then $z \neq 1$. By Theorem 2.5(a), we have

$$\text{D-spec}(\Gamma_H) = \Big\{[-2]^{\frac{1}{2}(n+1)z-3}, [\tfrac{z}{2}-2]^1, \left[\tfrac{1}{4}\left(2nz+z-8+z\sqrt{4n^2-12n+17}\right)\right]^1,$$
$$\left[\tfrac{1}{4}\left(2nz+z-8-z\sqrt{4n^2-12n+17}\right)\right]^1\Big\}.$$

We have $A_1 := |-2| = 2$; $A_2 := |\frac{z}{2}-2| = 2-\frac{z}{2}$ or $\frac{z}{2}-2$ according as $z=2,3$ or $z\geq 4$; $A_3 := \left|\frac{1}{4}\left(2nz+z-8+z\sqrt{4n^2-12n+17}\right)\right| = \frac{1}{4}\left(2nz+z-8+z\sqrt{4n^2-12n+17}\right)$ and

$$\begin{aligned} A_4 &:= \left|\frac{1}{4}\left(2nz+z-8-z\sqrt{4n^2-12n+17}\right)\right| \\ &= \begin{cases} -\frac{2nz+z-8-z\sqrt{4n^2-12n+17}}{4}, & \text{for } z=1,2 \\ \frac{2nz+z-8-z\sqrt{4n^2-12n+17}}{4}, & \text{for } z\geq 3. \end{cases} \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{E_D}(\Gamma_H) &= \left(\frac{(n+1)z}{2}-3\right)\times A_1 + 1\times A_2 + 1\times A_3 + 1\times A_4 \\ &= \begin{cases} 2n-3+\sqrt{4n^2-12n+17}, & \text{for } z=2 \\ 6n-5, & \text{for } z=3 \\ 2(nz+z-6), & \text{for } z\geq 4. \end{cases} \end{aligned}$$

By Theorem 2.5(a), we have $\text{DL-spec}(\Gamma_H) = \left\{[0]^1, \left[\frac{(n+1)z}{2}\right]^2, [nz]^{\frac{(n-1)z}{2}-1}, \left[\frac{(n+2)z}{2}\right]^{z-2}\right\}$ and $W(\Gamma_H) = \frac{1}{4}z\left(n^2z-2n+2z-2\right)$. Therefore, $\Delta(\Gamma_H) = \frac{n^2z-2n+2z-2}{n+1}$. We have

$$L_1 := \Big|0-\Delta(\Gamma_H)\Big| = \frac{n^2z-2n+2z-2}{n+1},$$

$$L_2 := \left|\frac{1}{2}(n+1)z-\Delta(\Gamma_H)\right| = -\frac{-n^2z+2nz+4n-3z+4}{2n+2},$$

$$L_3 := \left|nz - \Delta(\Gamma_H)\right| = \frac{n(z+2)-2z+2}{n+1}$$

and

$$L_4 := \left|\frac{1}{2}(n+2)z - \Delta(\Gamma_H)\right| = \begin{cases} \frac{-n^2z+3nz+4n-2z+4}{2n+2}, & \text{for } n=4 \text{ \& } z=2,3 \\ -\frac{-n^2z+3nz+4n-2z+4}{2n+2}, & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} \mathrm{E_{DL}}(\Gamma_H) &= 1 \times L_1 + 2 \times L_2 + \left(\frac{(n-1)z}{2} - 1\right) \times L_3 + (z-2) \times L_4 \\ &= \begin{cases} \frac{2\left(2n^2z-2n(z+3)+5z-6\right)}{n+1}, & \text{for } n=4 \text{ \& } z=2,3 \\ \frac{n^2z^2+2n^2z-3nz^2-2nz-4n+2z^2+2z-4}{n+1}, & \text{otherwise.} \end{cases} \end{aligned}$$

By Theorem 2.5(a), we also have $\text{DQ-spec}(\Gamma_H) = \Big\{[nz-4]^{\frac{(n-1)z}{2}-1}, \left[\frac{(n+2)z}{2}-4\right]^{z-2}$, $\left[\frac{(n+3)z}{2}-4\right]^1, \left[\frac{1}{4}\left(5nz+3z-16-z\sqrt{9n^2-34n+41}\right)\right]^1, \left[\frac{1}{4}\left(5nz+3z-16+z\sqrt{9n^2-34n+41}\right)\right]^1\Big\}$.

We have

$$B_1 := \left|(nz-4) - \Delta(\Gamma_H)\right| = \begin{cases} -\frac{nz-2n-2z-2}{n+1}, & \text{for } z=2 \text{ \& } n \geq 4; \\ & z=3 \text{ \& } n=4,6;\ z=4 \text{ \& } n=4 \\ = \frac{nz-2n-2z-2}{n+1}, & \text{otherwise,} \end{cases}$$

$$B_2 := \left|\left(\frac{(n+2)z}{2} - 4\right) - \Delta(\Gamma_H)\right| = \frac{n^2z-3nz+4n+2z+4}{2n+2},$$

$$B_3 := \left|\left(\frac{(n+3)z}{2} - 4\right) - \Delta(\Gamma_H)\right| = \frac{n^2z-4nz+4n+z+4}{2n+2},$$

$$\begin{aligned} B_4 :&= \left|\frac{1}{4}\left(-\sqrt{9n^2-34n+41}z + 5nz + 3z - 16\right) - \Delta(\Gamma_H)\right| \\ &= \begin{cases} \frac{n^2z-+8nz-8n-5z-8(nz+z)\sqrt{9n^2-34n+41}}{4(n+1)}, & \text{for } n=4 \text{ \& } z \geq 5 \\ -\frac{n^2z-+8nz-8n-5z-8(nz+z)\sqrt{9n^2-34n+41}}{4(n+1)}, & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} B_5 :&= \left|\frac{1}{4}\left(\sqrt{9n^2-34n+41}z + 5nz + 3z - 16\right) - \Delta(\Gamma_H)\right| \\ &= \frac{n^2z+8nz-8n-5z-8+(nz+z)\sqrt{9n^2-34n+41}}{4(n+1)}. \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{E_{DQ}}(\Gamma_H) &= \left(\frac{(n-1)z}{2} - 1\right) \times B_1 + (z-2) \times B_2 + 1 \times B_3 + 1 \times B_4 + 1 \times B_5 \\ &= \begin{cases} \frac{n^2z+n\left(\left(\sqrt{9n^2-34n+41}+8\right)z-8\right)+\left(\sqrt{9n^2-34n+41}-5\right)z-8}{2(n+1)}, & \text{for } z=2 \text{ \& } n \geq 4; \\ & z=3 \text{ \& } n=4,6; \\ & z=4 \text{ \& } n=4 \\ \frac{1}{5}\left(6z^2+9z-20\right), & \text{for } n=4 \text{ \& } z \geq 5 \\ \frac{z\left(n^2(2z-3)+n\left(\sqrt{9n^2-26n+33}-6z+4\right)+\sqrt{9n^2-26n+33}+4z+7\right)}{2(n+1)}, & \text{otherwise.} \end{cases} \end{aligned}$$

(b). If n is odd then, by Theorem 2.5(b), we have D-spec$(\Gamma_H) = \Big\{[-2]^{\frac{(n+1)z}{2}-2},$

$$\left[\tfrac{1}{2}(nz+z-4-z\sqrt{n^2-4n+7})\right]^1,\ \left[\tfrac{1}{2}(nz+z-4+z\sqrt{n^2-4n+7})\right]^1\Big\}.$$

We have

$$\begin{aligned} A_1' := &\left|\frac{1}{2}\left(nz+z-4-z\sqrt{n^2-4n+7}\right)\right| \\ = &\begin{cases} -\frac{1}{2}\left(nz+z-4-z\sqrt{n^2-4n+7}\right), & \text{for } z=1 \\ \frac{1}{2}\left(nz+z-4-z\sqrt{n^2-4n+7}\right), & \text{for } z\geq 2 \end{cases} \end{aligned}$$

and

$$A_2' := \left|\frac{1}{2}\left(nz+z-4+z\sqrt{n^2-4n+7}\right)\right| = \frac{1}{2}\left(nz+z-4+z\sqrt{n^2-4n+7}\right).$$

Hence

$$\begin{aligned} \mathrm{E_D}(\Gamma_H) &= \left(\frac{(n+1)z}{2}-2\right)\times|-2| + 1\times A_1' + 1\times A_2' \\ &= \begin{cases} \frac{1}{2}\left(4\sqrt{n^2-4n+7}+n-3\right), & \text{for } z=1 \\ \frac{5}{2}(nz+z-4), & \text{for } z\geq 2. \end{cases} \end{aligned}$$

By Theorem 2.5(b), we have DL-spec$(\Gamma_H) = \left\{[0]^1, \left[\frac{(n+1)z}{2}\right]^1, [nz]^{\frac{(n-1)z}{2}-1}, \left[\frac{(n+3)z}{2}\right]^{z-1}\right\}$ and $W(\Gamma_H) = \frac{1}{4}z\left(n^2z-2n+3z-2\right)$. Therefore, $\Delta(\Gamma_H) = \frac{n^2z-2n+3z-2}{n+1}$. We have

$$L_1' := \left|0-\Delta(\Gamma_H)\right| = \left|-\frac{n^2z-2n+3z-2}{n+1}\right| = \frac{n^2z-2n+3z-2}{n+1},$$

$$L_2' := \left|\frac{(n+1)z}{2}-\Delta(\Gamma_H)\right| = \begin{cases} \frac{-n^2z+2nz+4n-5z+4}{2n+2}, & \text{for } n=3,5 \ \&\ z=1 \\ -\frac{-n^2z+2nz+4n-5z+4}{2n+2}, & \text{otherwise,} \end{cases}$$

$$L_3' := \left|nz-\Delta(\Gamma_H)\right| = \left|\frac{nz+2n-3z+2}{n+1}\right| = \frac{nz+2n-3z+2}{n+1}$$

and

$$L_4' = \left|\frac{(n+3)z}{2}-\Delta(\Gamma_H)\right| = \begin{cases} \frac{-n^2z+4nz+4n-3z+4}{2n+2}, & \text{for } n=3 \ \&\ z\geq 1; \\ & n=5 \ \&\ z=1,2;\ n=7 \ \&\ z=1 \\ -\frac{-n^2z+4nz+4n-3z+4}{2n+2}, & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} \mathrm{E_{DL}}(\Gamma_H) &= 1\times L_1' + 1\times L_2' + \left(\frac{(n-1)z}{2}-1\right)L_3' + (z-1)\times L_4' \\ &= \begin{cases} \frac{2n^2z-4n+6z-4}{n+1}, & \text{for } n=3,5 \ \&\ z=1 \\ \frac{3n^2z-2nz-8n+11z-8}{n+1}, & \text{for } n=7 \ \&\ z=1;\ n=3 \ \&\ z\geq 2; \\ & n=5 \ \&\ z=2 \\ \frac{n^2z^2+2n^2z-4nz^2-2nz-4n+3z^2+4z-4}{n+1}, & \text{otherwise.} \end{cases} \end{aligned}$$

By Theorem 2.5(b), we also have $\text{DQ-spec}(\Gamma_H) = \Big\{[nz-4]^{\frac{(n-1)z}{2}-1}, \left[\frac{(n+3)z}{2}-4\right]^{z-1},$ $\left[\frac{1}{4}(5nz+5z-16-z\sqrt{9n^2-46n+73})\right]^1, \left[\frac{1}{4}(5nz+5z-16+z\sqrt{9n^2-46n+73})\right]^1\Big\}.$

We have

$$B_1' := \Big|(nz-4) - \Delta(\Gamma_H)\Big| = \begin{cases} -\frac{nz-2n-3z-2}{n+1}, & \text{for } z=1,2 \;\&n\geq 3;\; z=3 \;\&\; n=3,5,7,9;\\ & z=4,5 \;\&\; n=3,5; n=3 \;\&\; z\geq 1;\\ & n=5 \;\&\; 1\leq z\leq 5;\; n=7 \;\&\; z=1,2,3\\ \frac{nz-2n-3z-2}{n+1}, & \text{otherwise,} \end{cases}$$

$$B_2' := \left|\frac{1}{2}(n+3)z-4-\Delta(\Gamma_H)\right| = \frac{n^2z-4nz+4n+3z+4}{2n+2},$$

$$\begin{aligned} B_3' &:= \left|\frac{1}{4}\left(-\sqrt{9n^2-46n+73}z+5nz+5z-16\right)-\Delta(\Gamma_H)\right|\\ &= \begin{cases} \frac{n^2z+10nz-8n-7z-8-(nz+z)\sqrt{9n^2-46n+73}}{4(n+1)}, & \text{for } n=3 \;\&\; z\geq 2;\\ & n=5 \;\&\; z\geq 3;\; n=7 \;\&\; z\geq 56\\ -\frac{n^2z+10nz-8n-7z-8-(nz+z)\sqrt{9n^2-46n+73}}{4(n+1)}, & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} B_4' &:= \left|\frac{1}{4}\left(\sqrt{9n^2-46n+73}z+5nz+5z-16\right)-\Delta(\Gamma_H)\right|\\ &= \frac{n^2z+10nz-8n-7z-8+(nz+z)\sqrt{9n^2-46n+73}}{4(n+1)}. \end{aligned}$$

Hence

$$\begin{aligned} E_{DQ}(\Gamma_H) &= \left(\frac{(n-1)z}{2}-1\right)\times B_1' + (z-1)\times B_2' + 1\times B_3' + 1\times B_4'\\ &= \begin{cases} \frac{n^2z+10nz-8n-7z-8}{n+1}, & \text{for } n=3 \;\&\; z\geq 2;\\ & n=5 \;\&\; z=3,4,5\\ \frac{1}{2}\left(n+\sqrt{n(9n-46)+73}-\frac{16}{n+1}+9\right)z-4, & \text{for } n=3 \;\&\; z=1\\ \frac{(z-1)(n((n-4)z+4)+3z+4)}{n+1}, & \text{for } n=5 \;\&\; z\geq 6;\\ & n=7 \;\&\; z\geq 56\\ \frac{1}{2}\left(n+\sqrt{n(9n-46)+73}-\frac{16}{n+1}+9\right)z-4, & \text{for } n=5 \;\&\; z=1,2;\\ & n=7 \;\&\; z=1,2,3\\ \frac{1}{2}z\left(2\left(\frac{8}{n+1}-5\right)z+n(2z-3)+\sqrt{n(9n-46)+73}+9\right), & \text{otherwise.} \end{cases} \end{aligned}$$

□

Since the groups D_{2n} (where $n\geq 3$), $U_{(n,m)}$ (where $m\geq 3$ and $n\geq 2$), T_{4n} (where $n\geq 2$), SD_{8n} (where $n\geq 2$) and U_{6n} (where $n\geq 2$) belong to the class of groups considered in Theorem 3.4, we have the following corollaries.

Corollary 3.5. *Let H be the dihedral group D_{2n}, where $n\geq 3$.*

(a). *If* n *is odd then* $\mathrm{E_D}(\Gamma_H) = \frac{1}{2}\left(4\sqrt{n^2-4n+7}+n-3\right)$,

$$\mathrm{E_{DL}}(\Gamma_H) = \begin{cases} \frac{2(n-1)^2}{n+1}, & \textit{for } n = 3, 5 \\ \frac{3n^2-10n+3}{n+1}, & \textit{for } n \geq 7 \end{cases}$$

$$\textit{and } \mathrm{E_{DQ}}(\Gamma_H) = \begin{cases} 2, & \textit{for } n = 3 \\ \frac{1}{2}\left(\sqrt{9n^2-46n+73}+n-\frac{16}{n+1}+1\right), & \textit{for } n = 5,\ 7 \\ \frac{1}{2}\left(\sqrt{9n^2-46n+73}-n+\frac{16}{n+1}-1\right), & \textit{for } n \geq 9. \end{cases}$$

(b). *If* n *and* $\frac{n}{2}$ *are even then* $\mathrm{E_D}(\Gamma_H) = \sqrt{n^2-6n+17}+n-3$,

$$\mathrm{E_{DL}}(\Gamma_H) = \begin{cases} 4, & \textit{for } n = 4 \\ \frac{4\left(n^2-5n+4\right)}{n+2}, & \textit{for } n \geq 8 \end{cases}$$

$$\textit{and } \mathrm{E_{DQ}}(\Gamma_H) = \begin{cases} 4, & \textit{for } n = 4 \\ \frac{n^2+\left(\sqrt{9n^2-68n+164}+8\right)n+2\left(\sqrt{9n^2-68n+164}-18\right)}{2(n+2)}, & \textit{for } n \geq 8. \end{cases}$$

(c). *If* n *is even and* $\frac{n}{2}$ *is odd then* $\mathrm{E_D}(\Gamma_H) = \frac{5(n-2)}{2}$,

$$\mathrm{E_{DL}}(\Gamma_H) = \begin{cases} \frac{3n^2-12n+28}{n+2}, & \textit{for } n = 6, 10 \\ \frac{4\left(n^2-6n+8\right)}{n+2}, & \textit{otherwise} \end{cases}$$

$$\textit{and } \mathrm{E_{DQ}}(\Gamma_H) = \begin{cases} \frac{n^2+12n-44}{n+2}, & \textit{for } n = 6 \\ \frac{1}{2}\left(\sqrt{9n^2-92n+292}+n-\frac{64}{n+2}+10\right), & \textit{for } n = 10,\ 14 \\ \frac{1}{2}\left(\sqrt{9n^2-92n+292}+n+\frac{128}{n+2}-22\right), & \textit{otherwise.} \end{cases}$$

Corollary 3.6. *Let* H *be the group* $U_{(n,m)}$, *where* $m \geq 3$ *and* $n \geq 2$.

(a). *If* m *is odd then* $\mathrm{E_D}(\Gamma_H) = \frac{5}{2}(mn+n-4)$,

$$\mathrm{E_{DL}}(\Gamma_H) = \begin{cases} \frac{3m^2n-2mn-8m+11n-8}{m+1}, & \textit{for } m = 3 \,\&\, n \geq 2; m = 5 \,\&\, n = 2 \\ \frac{m^2n^2+2m^2n-4mn^2-2mn-4m+3n^2+4n-4}{m+1}, & \textit{otherwise} \end{cases}$$

$$\textit{and } \mathrm{E_{DQ}}(\Gamma_H) = \begin{cases} \frac{m^2n+10mn-8m-7n-8}{m+1}, & \textit{for } m = 3 \,\&\, n \geq 2; \\ & m = 5 \,\&\, n = 3, 4, 5 \\ \frac{(n-1)(m((m-4)n+4)+3n+4)}{m+1}, & \textit{for } m = 5 \,\&\, n \geq 6; \\ & m = 7 \,\&\, n \geq 56 \\ \frac{1}{2}\left(m+\sqrt{m(9m-46)+73}-\frac{16}{m+1}+9\right)n-4, & \textit{for } m = 5 \,\&\, n = 2; \\ & m = 7 \,\&\, n = 2, 3 \\ \frac{1}{2}n\left(2\left(\frac{8}{m+1}-5\right)n+m(2n-3)+\sqrt{m(9m-46)+73}+9\right), & \textit{otherwise.} \end{cases}$$

(b). *If m and $\frac{m}{2}$ are even then*

$$\mathrm{E_D}(\Gamma_H) = \begin{cases} 12(n-1), & \text{for } m = 4 \\ 2(mn+2n-6), & \text{for } m \geq 8, \end{cases}$$

$$\mathrm{E_{DL}}(\Gamma_H) = \begin{cases} 12(n-1), & \text{for } m = 4 \\ \frac{2m^2n(n+1)-4m\left(3n^2+n+1\right)+8\left(2n^2+n-1\right)}{m+2}, & \text{for } m \geq 8 \end{cases}$$

and $\mathrm{E_{DQ}}(\Gamma_H) =$

$$\begin{cases} 12(n-1), & \text{for } m = 4 \\ \frac{136}{5}, & \text{for } m = 8 \,\&\, n = 2 \\ \frac{1}{5}\left(24n^2+18n-20\right), & \text{for } m = 8 \,\&\, n \geq 3 \\ \frac{n\left(m^2(4n-3)+m\left(\sqrt{9m^2-52m+132}-24n+8\right)+2\left(\sqrt{9m^2-52m+132}+16n+14\right)\right)}{2(m+2)}, & \text{otherwise.} \end{cases}$$

(c). *If m is even and $\frac{m}{2}$ is odd then* $\mathrm{E_D}(\Gamma_H) = \frac{5}{2}(mn+2n-4)$,

$$\mathrm{E_{DL}}(\Gamma_H) = \begin{cases} \frac{3m^2n-4m(n+2)+44n-16}{m+2}, & \text{for } m = 6 \,\&\, n \geq 1; \\ \frac{2\left(m^2n(n+1)-2m\left(4n^2+n+1\right)+12n^2+8n-4\right)}{m+2}, & \text{otherwise} \end{cases}$$

and

$$\mathrm{E_{DQ}}(\Gamma_H) = \begin{cases} \frac{m^2n+4m(5n-2)-4(7n+4)}{m+2}, & \text{for } m = 6 \,\&\, n \geq 1; \\ & m = 10 \,\&\, n = 2 \\ \frac{(2n-1)\left(m^2n+m(4-8n)+12n+8\right)}{m+2}, & \text{for } m = 10 \,\&\, n \geq 3; \\ & m = 14 \,\&\, n \geq 28 \\ \frac{1}{2}n\left(\sqrt{9m^2-92m+292} - \frac{8(5m-6)n}{m+2} + m(4n-3)+18\right) & \text{otherwise.} \end{cases}$$

Corollary 3.7. *Let H be the group T_{4n}, where $n \geq 2$.*

(a). *If n is even then* $\mathrm{E_D}(\Gamma_H) = 2n-3+\sqrt{4n^2-12n+17}$,

$$\mathrm{E_{DL}}(\Gamma_H) = \begin{cases} 4, & \text{for } n = 2 \\ \frac{8n^2-20n+8}{n+1}, & \text{for } n \geq 4 \end{cases}$$

$$\text{and } \mathrm{E_{DQ}}(\Gamma_H) = \begin{cases} 4, & \text{for } n = 2 \\ \frac{n^2+\left(\sqrt{9n^2-34n+41}+4\right)n+\sqrt{9n^2-34n+41}-9}{n+1}, & \text{for } n \geq 4. \end{cases}$$

(b). *If n is odd then* $\mathrm{E_D}(\Gamma_H) = 5(n-1)$,

$$\mathrm{E_{DL}}(\Gamma_H) = \begin{cases} \frac{6n^2-12n+14}{n+1}, & \text{for } n = 3, 5 \\ \frac{8n^2-24n+16}{n+1}, & \text{otherwise} \end{cases}$$

$$\text{and } \mathrm{E_{DQ}}(\Gamma_H) = \begin{cases} 8, & \text{for } n = 3 \\ 2\sqrt{17}+\frac{22}{3}, & \text{for } n = 5 \\ 8\sqrt{3}+10, & \text{for } n = 7 \\ \sqrt{9n^2-46n+73}+n+\frac{32}{n+1}-11, & \text{otherwise.} \end{cases}$$

Corollary 3.8. *Let H be the semidihedral group SD_{8n}, where $n \geq 2$.*

(a). *If n is even then* $\mathrm{E_D}(\Gamma_H) = \sqrt{16n^2 - 24n + 17} + 4n - 3$,

$$\mathrm{E_{DL}}(\Gamma_H) = \begin{cases} \frac{56}{5}, & \text{for } n = 2 \\ \frac{32n^2-40n+8}{2n+1}, & \text{otherwise} \end{cases}$$

$$\text{and } \mathrm{E_{DQ}}(\Gamma_H) = \frac{4n^2 + 2\left(\sqrt{36n^2 - 68n + 41} + 4\right)n + \sqrt{36n^2 - 68n + 41} - 9}{2n+1}.$$

(b). *If n is odd then* $\mathrm{E_D}(\Gamma_H) = 10n$,

$$\mathrm{E_{DL}}(\Gamma_H) = \begin{cases} 24, & \text{for } n = 3 \\ \frac{24n^2-76n+60}{n+1}, & \text{otherwise} \end{cases}$$

$$\text{and } \mathrm{E_{DQ}}(\Gamma_H) = \begin{cases} \frac{4n^2+32n-36}{n+1}, & \text{for } n = 3,\ 5 \\ 2\left(\sqrt{9n^2 - 46n + 73} + 5n + \frac{64}{n+1} - 31\right), & \text{otherwise.} \end{cases}$$

Corollary 3.9. *Let H be the group U_{6n}, where $n \geq 2$. Then $\mathrm{E_D}(\Gamma_H) = 10(n-1)$ and $\mathrm{E_{DL}}(\Gamma_H) = \mathrm{E_{DQ}}(\Gamma_H) = 8(n-1)$.*

We conclude this section with the following result.

Theorem 3.10. *Let H be the group V_{8n}, where $n \geq 2$.*

(a). *If n is even then* $\mathrm{E_D}(\Gamma_H) = 4(2n-1)$,

$$\mathrm{E_{DL}}(\Gamma_H) = \begin{cases} 12, & \text{for } n = 2 \\ \frac{12\left(2n^2-5n+3\right)}{n+1}, & \text{for } n \geq 4 \end{cases}$$

$$\text{and } \mathrm{E_{DQ}}(\Gamma_H) = \begin{cases} 12, & \text{for } n = 2 \\ \frac{136}{5}, & \text{for } n = 4 \\ \frac{2\left(5n^2-20n+(n+1)\sqrt{n(9n-34)+41}+23\right)}{n+1}, & \text{for } n \geq 6. \end{cases}$$

(b). *If n is odd then* $\mathrm{E_D}(\Gamma_H) = \sqrt{16n^2 - 24n + 17} + 4n - 3$,

$$\mathrm{E_{DL}}(\Gamma_H) = \frac{8\left(4n^2 - 5n + 1\right)}{2n+1}$$

$$\text{and } \mathrm{E_{DQ}}(\Gamma_H) = \frac{4n^2 + 2\left(\sqrt{36n^2 - 68n + 41} + 4\right)n + \sqrt{36n^2 - 68n + 41} - 9}{2n+1}.$$

Proof. (a). If n is even then, by Theorem 2.13(a), we have $\text{D-spec}(\Gamma_H) = \left\{[0]^1, [-2]^{2n-1}, \left[2n - 1 - \sqrt{4n^2 - 12n + 17}\right]^1, \left[2n - 1 + \sqrt{4n^2 - 12n + 17}\right]^1\right\}$.

We have

$$A_1 := \left|2n-1-\sqrt{4n^2-12n+17}\right| = 2n-1-\sqrt{4n^2-12n+17}$$

and

$$A_2 := \left|2n-1+\sqrt{4n^2-12n+17}\right| = 2n-1+\sqrt{4n^2-12n+17}.$$

Hence

$$\mathrm{E_D}(\Gamma_H) = 1\times|0| + (2n-1)\times|-2| + 1\times A_1 + 1\times A_2 = 4(2n-1).$$

By Theorem 2.13(a), we have $\text{DL-spec}(\Gamma_H) = \left\{[0]^1, [2n+2]^2, [4n]^{2n-3}, [2n+4]^2\right\}$ and $W(\Gamma_H) = \frac{1}{2}(8n^2-4n+12)$. Therefore, $\Delta(\Gamma_H) = \frac{4n^2-2n+6}{n+1}$. We have

$$L_1 := \left|0-\Delta(\Gamma_H)\right| = \frac{4n^2-2n+6}{n+1},$$

$$L_2 := \left|(2n+2)-\Delta(\Gamma_H)\right| = \frac{2\left(n^2-3n+2\right)}{n+1}, \quad L_3 := \left|4n-\Delta(\Gamma_H)\right| = \frac{6(n-1)}{n+1}$$

and

$$L_4 := \left|(2n+4)-\Delta(\Gamma_H)\right| = \begin{cases} 2, & \text{for } n=2 \\ \frac{2\left(n^2-4n+1\right)}{n+1}, & \text{for } n\geq 4. \end{cases}$$

Hence

$$\begin{aligned} \mathrm{E_{DL}}(\Gamma_H) &= 1\times L_1 + 2\times L_2 + (2n-3)\times L_3 + 2\times L_4 \\ &= \begin{cases} 12, & \text{for } n=2 \\ \frac{12\left(2n^2-5n+3\right)}{n+1}, & \text{for } n\geq 4. \end{cases} \end{aligned}$$

By Theorem 2.13(a), we also have $\text{DQ-spec}(\Gamma_H) = \big\{[4n-4]^{2n-3}, [2n]^2, [2n+2]^1,$ $\big[-\sqrt{9n^2-34n+41}+5n-1\big]^1, \big[\sqrt{9n^2-34n+41}+5n-1\big]^1\big\}$.

We have

$$B_1 := \left|(4n-4)-\Delta(\Gamma_H)\right| = \begin{cases} -\frac{2(n-5)}{n+1}, & \text{for } n=2,4 \\ \frac{2(n-5)}{n+1}, & \text{for } n\geq 6, \end{cases}$$

$$B_2 := \left|2n-\Delta(\Gamma_H)\right| = \frac{2\left(n^2-2n+3\right)}{n+1}, B_3 := \left|(2n+2)-\Delta(\Gamma_H)\right| = \frac{2\left(n^2-3n+2\right)}{n+1},$$

$$B_4 := \left|5n-1-\sqrt{9n^2-34n+41}-\Delta(\Gamma_H)\right| = -\frac{n^2+6n-7-(n+1)\sqrt{9n^2-34n+41}}{n+1}$$

and

$$B_5 := \left|\sqrt{9n^2-34n+41}+5n-1-\Delta(\Gamma_H)\right| = \frac{n^2+6n-7+(n+1)\sqrt{9n^2-34n+41}}{n+1}.$$

Hence

$$\begin{aligned} \mathrm{E}_{\mathrm{DQ}}(\Gamma_H) &= (2n-3)\times B_1 + 2\times B_2 + 1\times B_3 + 1\times B_4 + 1\times B_5 \\ &= \begin{cases} 12, & \text{for } n=2 \\ \frac{136}{5}, & \text{for } n=4 \\ \frac{2\left(5n^2-20n+(n+1)\sqrt{n(9n-34)+41}+23\right)}{n+1}, & \text{for } n\geq 6. \end{cases} \end{aligned}$$

(b). If n is odd, then by [38, Proposition 2.4], we have $\Gamma_H = K_{2n-1,1,1} = \Gamma_{D_{2\times 4n}}$. Hence, the result follows from Corollary 2.6. □

4. Comparing different distance energies

Motivated by Problem 1.1 – Problem 1.3, in this section, we compare the distance energy, distance Laplacian energy and distance signless Laplacian energy of NCCC-graphs for the finite non-abelian groups discussed in the previous sections. We choose graphical methods to compare various distance energies of NCCC-graphs. The following figures describe the relations among $\mathrm{E}_{\mathrm{D}}(\Gamma_H)$, $\mathrm{E}_{\mathrm{DL}}(\Gamma_H)$ and $\mathrm{E}_{\mathrm{DQ}}(\Gamma_H)$ for the groups $H = D_{2n}$, T_{4n}, SD_{8n}, U_{6n} and V_{8n}.

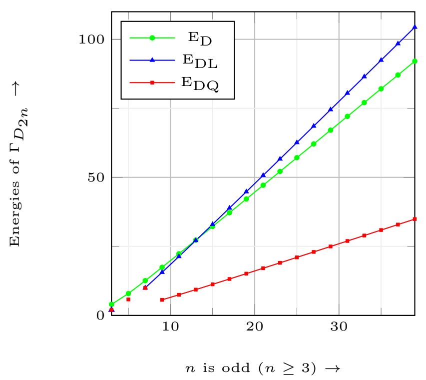

Figure 1: **Energies of $\Gamma_{D_{2n}}$, n is odd**

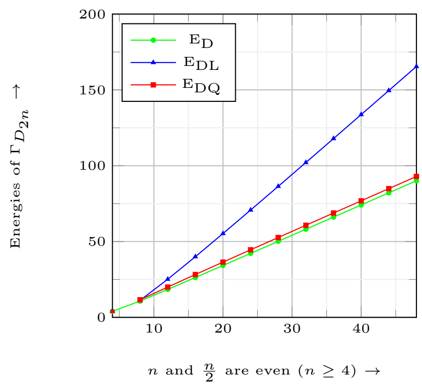

Figure 2: **Energies of $\Gamma_{D_{2n}}$, n and $\frac{n}{2}$ are even**

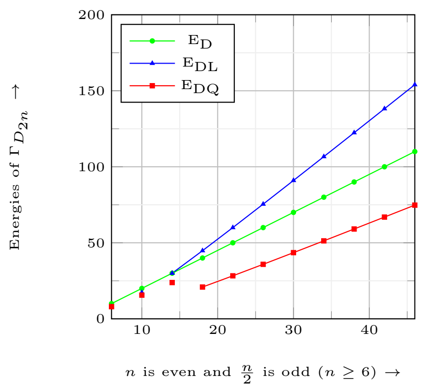

Figure 3: **Energies of $\Gamma_{D_{2n}}$, n is even and $\frac{n}{2}$ is odd**

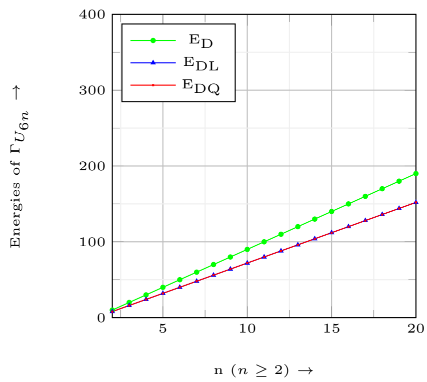

Figure 4: **Energies of $\Gamma_{U_{6n}}$**

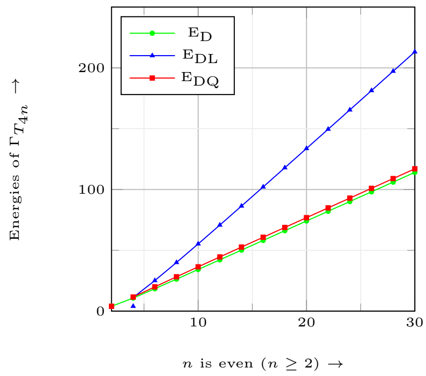

Figure 5: **Energies of $\Gamma_{T_{4n}}$, n is even**

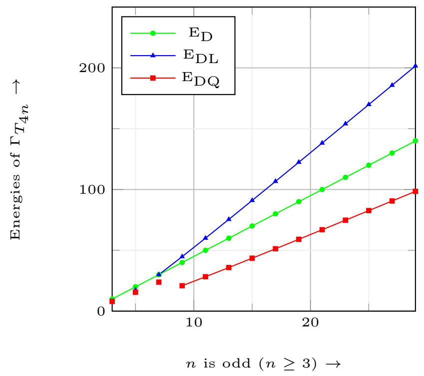

Figure 6: **Energies of $\Gamma_{T_{4n}}$, n is odd**

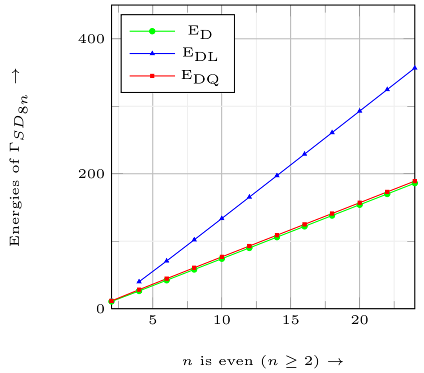

Figure 7: **Energies of $\Gamma_{SD_{8n}}$, n is even**

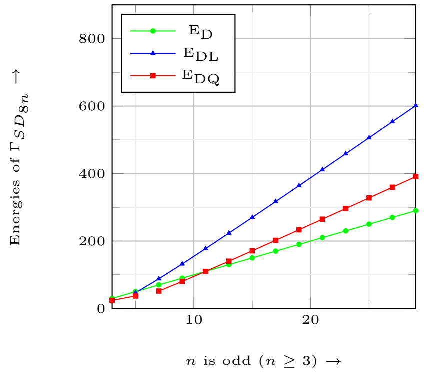

Figure 8: **Energies of $\Gamma_{SD_{8n}}$, n is odd**

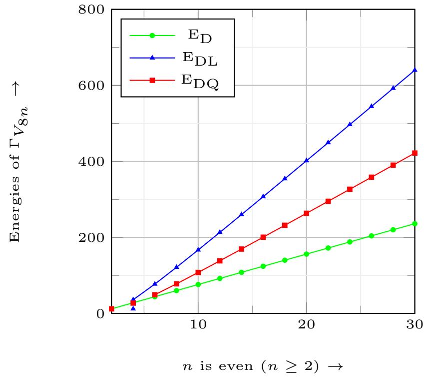

Figure 9: **Energies of $\Gamma_{V_{8n}}$, n is even**

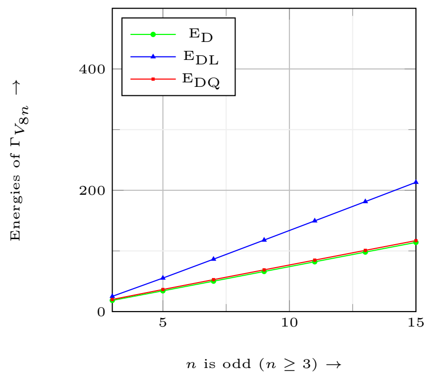

Figure 10: **Energies of $\Gamma_{V_{8n}}$, n is odd**

By observing the Figures 1 – 10 we get the following result.

Theorem 4.1. *Let $H = D_{2n}$ (where $n \geq 3$), T_{4n} (where $n \geq 2$), SD_{8n} (where $n \geq 2$), V_{8n} (where $n \geq 2$) and U_{6n} (where $n \geq 2$). Then*

(a). $\mathrm{E_D}(\Gamma_H) = \mathrm{E_{DL}}(\Gamma_H) = \mathrm{E_{DQ}}(\Gamma_H)$ *if and only if $H \cong D_8$ or T_8 or V_{16};*

(b). $\mathrm{E_{DL}}(\Gamma_H) = \mathrm{E_{DQ}}(\Gamma_H) < \mathrm{E_D}(\Gamma_H)$ *if and only if $H \cong D_6$ or D_{12} or T_{12} or SD_{24} or U_{6n} ($n \geq 2$);*

(c). $\mathrm{E_{DL}}(\Gamma_H) < \mathrm{E_{DQ}}(\Gamma_H) < \mathrm{E_D}(\Gamma_H)$ *if and only if* $H \cong D_{10}$*;*

(d). $\mathrm{E_{DQ}}(\Gamma_H) < \mathrm{E_{DL}}(\Gamma_H) < \mathrm{E_D}(\Gamma_H)$ *if and only if* $H \cong D_{14}$ *or* D_{18} *or or* D_{20} *or* D_{22} *or* D_{26} *or* T_{20} *or* SD_{40}*;*

(e). $\mathrm{E_{DQ}}(\Gamma_H) < \mathrm{E_D}(\Gamma_H) < \mathrm{E_{DL}}(\Gamma_H)$ *if and only if* $H \cong D_{2n}$ *(*n *is odd and* $n \geq 15$*;* n *is even,* $\frac{n}{2}$ *is odd and* $n \geq 18$*) or* T_{4n} *(*n *is odd and* $n \geq 9$*) or* SD_{56} *or* SD_{72} *or* SD_{88} *or* v_{32}*;*

(f). $\mathrm{E_D}(\Gamma_H) < \mathrm{E_{DL}}(\Gamma_H) < \mathrm{E_{DQ}}(\Gamma_H)$ *if and only if* $H \cong D_{16}$ *or* T_{16} *or* SD_{16}*;*

(g). $\mathrm{E_D}(\Gamma_H) < \mathrm{E_{DQ}}(\Gamma_H) < \mathrm{E_{DL}}(\Gamma_H)$ *if and only if* $H \cong D_{2n}$ *(*n*,* $\frac{n}{2}$ *are even and* $n \geq 12$*) or* T_{4n} *(*n *is even and* $n \geq 6$*) or* SD_{8n} *(*n *is even and* $n \geq 4$*;* n *is odd and* $n \geq 13$*) or* V_{8n} *(*n *is even and* $n \geq 6$*;* n *is odd);*

(h). $\mathrm{E_{DQ}}(\Gamma_H) < \mathrm{E_D}(\Gamma_H) = \mathrm{E_{DL}}(\Gamma_H)$ *if and only if* $H \cong D_{28}$ *or* T_{28}*.*

We conclude this paper with the following corollary related to Problem 1.2.

Corollary 4.2. *Let* $H = D_{2n}$ *(where* $n \geq 3$*),* T_{4n} *(where* $n \geq 2$*),* SD_{8n} *(where* $n \geq 2$*),* V_{8n} *(where* $n \geq 2$*) and* U_{6n} *(where* $n \geq 2$*). Then* $\mathrm{E_{DL}}(\Gamma_H) = \mathrm{E_{DQ}}(\Gamma_H)$ *if and only if* $H \cong D_6, D_8$*,* D_{12} *or* T_8*,* T_{12} *or* V_{16}*,* SD_{24} *or* U_{6n} *(*$n \geq 2$*).*

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