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A note on injective dimension of local cohomology modules

Research Article

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Abstract: In this study, we assume that R is a commutative Noetherian ring with nonzero identity. We present upper bounds for the injective dimension of I, where I is any ideal in the ring R, in terms of the injective dimension of its local cohomology modules and an upper bound for the injective dimension that involves the theory of local cohomology modules. Since I is an ideal in R, we obtain applications of the theory in a general context.

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1. Introduction

Throughout the paper, R is a commutative Noetherian ring with non-zero identity.

For an R-module M, we denote $\mathrm{id}_R(M)$ as the injective dimension of M. We also write $\mathrm{H}^i_{\mathfrak{a}}(M)$, for $\mathfrak{a}$ an ideal of R, the i-th local cohomology module of M with respect to $\mathfrak{a}$. For basic results, notations, and terminologies not given in this paper, the reader is referred to [2], [3] and [8].

As applications, we put some results on injective dimension of local cohomology modules. Recall that the cohomological dimension of M with respect to $\mathfrak{a}$, denoted by $\mathrm{cd}_R(\mathfrak{a}, M)$, is the largest integer i in which $\mathrm{H}^i_{\mathfrak{a}}(M)$ is not zero, according to [5].

In Section 2, we put some definitions and prerequisites for a better understanding of the theory and we put some results.

We prove results on the injective dimension of local cohomology modules with respect to the theory in question.

Here, we use the properties of commutative algebra and homological algebra for the development of the results (see [1] and [8]).

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2. Main results on injective dimension

In this section, we present some results on the injective dimension of local cohomology modules.

Definition 2.1. The injective dimension of an R-module M is the length of the shortest injective resolution of M.

Let M be an arbitrary R–module. We denote

$$\mathrm{H}^i_{\mathfrak{a}}(M, I) = \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R\left(M/\mathfrak{a}^n M, I\right),$$

as the i-th generalized local cohomology module of M and I with respect to $\mathfrak{a}$, according to [6]. We have the isomorphisms

$$\mathrm{H}^i_{\mathfrak{a}}(M, I) \cong \mathrm{H}^i\left(\mathrm{Hom}_R(M, \Gamma_{\mathfrak{a}}(E^\bullet_I))\right)$$

where we have that $E^\bullet_I$ is a deleted injective resolution of I (see [4, Lemma 2.1(i)]). Thus, $\Gamma_{\mathfrak{a}}(M, I) \cong \mathrm{Hom}_R(M, \Gamma_{\mathfrak{a}}(I))$ and if

$$\mathrm{Supp}_R(M) \bigcap \mathrm{Supp}_R(I) \subseteq \mathrm{Var}(\mathfrak{a}),$$

then $\mathrm{H}^i_{\mathfrak{a}}(M, I) \cong \mathrm{Ext}^i_R(M, I)$, where M is any R-module.

Here, we denote $\mathrm{Var}(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{a} \subseteq \mathfrak{p}\}$.

Lemma 2.2. *Let I be an ideal in R. Suppose that N is a finitely generated R–module. Assume also that $\bar{I} = I/\Gamma_{\mathfrak{a}}(I)$ and $Q = E_R(\bar{I})/\bar{I}$, where $E_R(\bar{I})$ is an injective hull of $\bar{I}$. Then the following statements are true.*

(1) $\mathrm{H}^i_{\mathfrak{a}}(Q) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(I)$ *for all* $i \geq 0$.

(2) $\mathrm{H}^i_{\mathfrak{a}}(N, Q) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(N, \bar{I})$ *for all* $i \geq 0$.

Proof. Since $\Gamma_{\mathfrak{a}}(\bar{I}) = 0$, $\Gamma_{\mathfrak{a}}(E_R(\bar{I})) = 0$ and so

$$\Gamma_{\mathfrak{a}}(N, E_R(\bar{I})) \cong \mathrm{Hom}_R(N, \Gamma_{\mathfrak{a}}(E_R(\bar{I}))) = 0.$$

Applying the derived functors of $\Gamma_{\mathfrak{a}}(\bullet)$ and $\Gamma_{\mathfrak{a}}(N, \bullet)$ to the short exact sequence

$$0 \to \bar{I} \to E_R(\bar{I}) \to Q \to 0$$

we obtain the isomorphisms

$$\mathrm{H}^i_{\mathfrak{a}}(Q) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(\bar{I}) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(I)$$

and

$$\mathrm{H}^i_{\mathfrak{a}}(N, Q) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(N, \bar{I})$$

for all $i \geq 0$. $\square$

The following lemma is crucial to prove the main results of this paper. We observe, by convention, that $\mathrm{id}_R(0) = -\infty$.

Lemma 2.3. *Let I be an ideal in R. Let s be a non-negative integer, such that*

$$\sup\left\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(I)) + i \mid i \geq 0\right\} < s,$$

and N a finitely generated R–module. Then $\mathrm{H}^s_{\mathfrak{a}}(N, I) = 0$.

Proof. We prove this by induction on s. The case $s = 0$ is clear. Suppose that $s > 0$ and that $s - 1$ is settled. Let $\bar{I} = I/\Gamma_{\mathfrak{a}}(I)$ and $Q = E_R(\bar{I})/\bar{I}$. Since

$$\sup\left\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(I)) + i \mid i \geq 0\right\} < s,$$

$\mathrm{id}_R(\Gamma_{\mathfrak{a}}(I)) < s$ and

$$\sup\left\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(I)) + i \mid i \geq 1\right\} < s.$$

Thus $\mathrm{H}^s_{\mathfrak{a}}(N, \Gamma_{\mathfrak{a}}(I)) = 0$ and

$$\sup\left\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(Q)) + i \mid i \geq 0\right\} < s - 1$$

from Lemma 2.2(1). Hence, by the induction hypothesis on Q, we have $\mathrm{H}^{s-1}_{\mathfrak{a}}(N, Q) = 0$ and so, by Lemma 2.2(2), $\mathrm{H}^s_{\mathfrak{a}}(N, \bar{I}) = 0$. Now, by the short exact sequence

$$0 \to \Gamma_{\mathfrak{a}}(I) \to I \to \bar{I} \to 0$$

we get the long exact sequence

$$\ldots \to \mathrm{H}^s_{\mathfrak{a}}(N, \Gamma_{\mathfrak{a}}(I)) \to \mathrm{H}^s_{\mathfrak{a}}(N, I) \to \mathrm{H}^s_{\mathfrak{a}}(N, \bar{I}) \to \ldots$$

which shows that $\mathrm{H}^s_{\mathfrak{a}}(N, I) = 0$. □

For the R–module I, we say that $\mathrm{id}_R(\mathfrak{a}, I) \leq n$ if,

$$\mathrm{Ext}^s_R(R/\mathfrak{p}, I) = 0,$$

for all $\mathfrak{p} \in \mathrm{Var}(\mathfrak{a})$ and for all $s > n$.

We call $\mathrm{id}_R(\mathfrak{a}, I)$ the $\mathfrak{a}$–relative injective dimension of I (see [7]).

Corollary 2.4. *Let I be an ideal in R. For the R-module I, let $\mathfrak{b}$ be an ideal of R such that $\mathrm{Var}(\mathfrak{b}) \subseteq \mathrm{Var}(\mathfrak{a})$. Then*

$$\mathrm{id}_R(\mathfrak{b}, I) \leq \sup\left\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(I)) + i \mid i \geq 0\right\}.$$

In particular,

$$\mathrm{id}_R(\mathfrak{a}, I) \leq \sup\left\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(I)) + i \mid i \geq 0\right\}.$$

Proof. Assume that the right-hand side of the inequality is a finite number, say n. Assume also that $\mathfrak{p} \in \mathrm{Var}(\mathfrak{b})$ and $s > n$. Take $N = R/\mathfrak{p}$ in Lemma 2.3 to get $\mathrm{H}^s_{\mathfrak{a}}(R/\mathfrak{p}, I) = 0$. The assertion follows because we have the isomorphism $\mathrm{Ext}^s_R(R/\mathfrak{p}, I) \cong \mathrm{H}^s_{\mathfrak{a}}(R/\mathfrak{p}, I)$. □

The following theorem is the first main result of this paper and, among other things, shows that, in the case where we have a maximal ideal $\mathfrak{m}$ of R, if $\mathrm{id}_R(I) = \infty$, then for the ideal $\mathfrak{m}$ of R there exists an integer t such that $\mathrm{id}_R(\mathrm{H}^t_{\mathfrak{m}}(I)) = \infty$.

Theorem 2.5. *Let $(R, \mathfrak{m})$ be a local ring, and I ideal in R. Suppose that I is a R-module finitely generated. Then*

$$\mathrm{id}_R(I) \leq \sup\left\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{m}}(I)) + i \mid i \geq 0\right\}.$$

In particular, if $\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{m}}(I)) < \infty$ for all $i \geq 0$, then $\mathrm{id}_R(I) < \infty$.

Proof. Since $\mathrm{id}_R(I) = \mathrm{id}_R(\mathfrak{m}, I)$ (see [7]), the assertion follows from Corollary 2.4. For the last part, note that $\mathrm{cd}_R(\mathfrak{m}, I)$ is finite from [2, Theorem 3.3.1 or Theorem 6.1.2] and we have

$$\sup\left\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{m}}(I))+i \mid i\geq 0\right\}=\sup\left\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{m}}(I))+i \mid 0\leq i\leq \mathrm{cd}_R(\mathfrak{m},I)\right\},$$

because for all $i>\mathrm{cd}_R(\mathfrak{m},I)$, $\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{m}}(I))+i=-\infty+i=-\infty$ (see [2, Conventions 11.1.8]). □

It is known that if $(S,\mathfrak{m})$ is a Cohen-Macaulay local ring and $\mathrm{H}^{\dim(S)}_{\mathfrak{m}}(S)\cong E_S(S/\mathfrak{m})$, then S is Gorenstein. That means S is Gorenstein whenever $\mathrm{H}^{\dim(S)}_{\mathfrak{m}}(S)\cong E_S(S/\mathfrak{m})$ and $\mathrm{H}^i_{\mathfrak{m}}(S)=0$ for all $i\neq\dim(S)$. In the following corollary, we prove that R is Gorenstein if $\mathrm{H}^i_{\mathfrak{a}}(R)$ has finite injective dimension for all $i\geq 0$, where we have that R is not local and $\mathfrak{a}$ is maximal ideal of R.

Corollary 2.6. *Let R be as before, I ideal in R. Suppose that $\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(R))<\infty$ for all $i\geq 0$, where $\mathfrak{a}$ is any ideal in R. Then R is Gorenstein.*

Proof. Let $\mathfrak{p}$ be a prime ideal of R. By assumption, we have $\mathrm{id}_{R_{\mathfrak{p}}}(\mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}}(R_{\mathfrak{p}}))<\infty$ for all $i\geq 0$. Thus, we get $\mathrm{id}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}})<\infty$ from Theorem 2.5. Therefore, $R_{\mathfrak{p}}$ is a local Gorenstein ring. Hence R is Gorenstein, as we desired. □

We need the following lemma to prove the second main result of this paper.

Lemma 2.7. *Let I be an ideal in R. Let s,t be non-negative integers, such that*

$$\sup\left\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(I))-t+i-1 \mid i<t\right\}\bigcup A\bigcup B<s,$$

where $\mathfrak{a}$ ideal in R and $A=\{\mathrm{id}_R(I)-t\}$ and

$$B=\left\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(I))-t+i+1 \mid i>t\right\}.$$

Suppose that N is a finitely generated R-module. Then, we have that

$$\mathrm{Ext}^s_R(N,\mathrm{H}^t_{\mathfrak{a}}(I))=0.$$

Proof. Let $\bar{I}=I/\Gamma_{\mathfrak{a}}(I)$ and $Q=E_R(\bar{I})/\bar{I}$. We prove this by induction on t. Let $t=0$. Since $\mathrm{id}_R(I)<s$ and

$$\sup\left\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(I))+i+1 \mid i>0\right\}<s,$$

$\mathrm{H}^s_{\mathfrak{a}}(N,I)=0$ and

$$\sup\left\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(\bar{I}))+i \mid i\geq 0\right\}<s-1.$$

Thus $\mathrm{H}^{s-1}_{\mathfrak{a}}(N,\bar{I})=0$ from Lemma 2.3. Hence by the long exact sequence

$$\ldots\to\mathrm{H}^{s-1}_{\mathfrak{a}}(N,\bar{I})\to\mathrm{H}^s_{\mathfrak{a}}(N,\Gamma_{\mathfrak{a}}(I))\to\mathrm{H}^s_{\mathfrak{a}}(N,I)\to\ldots,$$

we get

$$\mathrm{Ext}^s_R(N,\Gamma_{\mathfrak{a}}(I))\cong\mathrm{H}^s_{\mathfrak{a}}(N,\Gamma_{\mathfrak{a}}(I))=0.$$

Suppose that $t>0$ and that $t-1$ is settled. From the exact sequences

$$0\to\Gamma_{\mathfrak{a}}(I)\to I\to\bar{I}\to 0$$

and

$$0\to\bar{I}\to E_R(\bar{I})\to Q\to 0,$$

we get $\mathrm{id}_R(Q)<s+t-1$ because we have $\mathrm{id}_R(\Gamma_{\mathfrak{a}}(I))<s+t+1$ and $\mathrm{id}_R(I)<s+t$ by assumptions. Thus

$$\sup\left\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(I))-t+i-1 \mid 0<i<t\right\}\bigcup C\bigcup D<s,$$

where $C = \{\mathrm{id}_R(Q) - (t-1)\}$ and $D = \left\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(I)) - t + i + 1 \mid i > t\right\}$, and thus, by Lemma 2.2(1),

$$\sup\left\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(Q)) - (t-1) + i - 1 \mid i < t-1\right\} \bigcup C \bigcup \mathrm{E} < s,$$

where $\mathrm{E} = \left\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(Q)) - (t-1) + i + 1 \mid i > t-1\right\}$.

Now, from the induction hypothesis on Q, we have $\mathrm{Ext}^s_R(N, \mathrm{H}^{t-1}_{\mathfrak{a}}(Q)) = 0$. Therefore, again by Lemma 2.2(1), we have

$$\mathrm{Ext}^s_R(N, \mathrm{H}^t_{\mathfrak{a}}(I)) = 0.$$

□

Now, we can state our second main result.

Theorem 2.8. *Let I be an ideal in R. Let t be a non-negative integer, and the ideal $\mathfrak{a}$ of R. Then*

$$\mathrm{id}_R(\mathrm{H}^t_{\mathfrak{a}}(I)) \leq \sup\left\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(I)) - t + i - 1 \mid i < t\right\} \bigcup A \bigcup B,$$

where $A = \{\mathrm{id}_R(I) - t\}$ and $B = \left\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(I)) - t + i + 1 \mid i > t\right\}$.

Proof. Assume that the right-hand side of the inequality is a finite number, say n. Assume also that $s > n$ and N is a finitely generated R–module. By Lemma 2.7, $\mathrm{Ext}^s_R(N, \mathrm{H}^t_{\mathfrak{a}}(I)) = 0$. Thus, the assertion follows. □

The following results are immediate applications of the above theorems.

Corollary 2.9. *Let I be an ideal in R. Suppose that I is a finitely generated R–module, and t is a non-negative integer such that $\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(I)) < \infty$ for all $i \neq t$, where $\mathfrak{a}$ ideal in R. Then $\mathrm{id}_R(\mathrm{H}^t_{\mathfrak{a}}(I)) < \infty$ if and only if $\mathrm{id}_R(I) < \infty$.*

Proof. This follows from Theorems 2.5 and 2.8. □

Corollary 2.10. *Let I be an ideal in R. Let t be a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(I)$ is injective for all $i \neq t$, where $\mathfrak{a}$ ideal in R. Then, we have that*

$$\mathrm{id}_R(\mathrm{H}^t_{\mathfrak{a}}(I)) \leq \mathrm{id}_R(I) - t + 1.$$

Proof. By Theorem 2.8, we have

$$\mathrm{id}_R(\mathrm{H}^t_{\mathfrak{a}}(I)) \leq \sup\left\{\mathrm{id}_R(I) - t, \mathrm{cd}_R(\mathfrak{a}, I) - t + 1\right\}.$$

Thus the assertion follows because $\mathrm{cd}_R(\mathfrak{a}, I) \leq \mathrm{id}_R(I)$. □

We finish the article with one more corollary.

Corollary 2.11. *Let R be as before, I ideal in R. Suppose that I is a finitely generated R–module, such that $\mathrm{H}^i_{\mathfrak{a}}(I)$ is injective for all $i < \mathrm{cd}_R(\mathfrak{a}, I)$, where $\mathfrak{a}$ is ideal in R. Then*

$$\mathrm{id}_R(I) - \mathrm{cd}_R(\mathfrak{a}, I) \leq \mathrm{id}_R(\mathrm{H}^{\mathrm{cd}_R(\mathfrak{a},I)}_{\mathfrak{a}}(I)) \leq \mathrm{id}_R(I) - \mathrm{cd}_R(\mathfrak{a}, I) + 1.$$

Proof. It follows from Theorem 2.5 and Corollary 2.10. □

3. Conclusion

With the results of this work, we find applications for the theory of local cohomology within of the theory of commutative algebra.

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References

[1] M.F. Atiyah, I.G. Macdonald, Introduction to commutative algebra, University of Oxford, 1969.
[2] M. P. Brodmann, R. Y. Sharp, Local Cohomology: An algebraic introduction with geometric applications, Cambridge, Cambridge University Press, 1998.
[3] W. Bruns, J. Herzog, Cohen-Macaulay rings, Cambridge, Cambridge University Press, 1998.
[4] K. Divaani-Aazar, R. Sazeedeh, M. Tousi, On vanishing of generalized local cohomology modules, Algebra Colloq. 12 (2005).
[5] R. Hartshorne, Cohomological dimension of algebraic varieties, Ann. of Math. 88 (1968) 403 - 450.
[6] J. Herzog, Komplexe, auflösungen und dualität in der lokalen algebra, Habilitationss-chrift, Universitat Regensburg, 1970.
[7] A. R. Naghipour, Some results on cofinite modules, International Electronic Journal of Algebra 11 (2012) 82 - 95.
[8] J.J. Rotman, An introduction to homological algebra, University of Illinois, Urbana, Academic Press, 1979.