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Combinatorial properties of certain Toeplitz matrices

Research Article

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Abstract: In additive combinatorics, a family of finite sets $\{\mathcal{A}_i\}$ is said to have bounded doubling if there exists a uniform constant K such that $|\mathcal{A}_i + \mathcal{A}_i| \leq K|\mathcal{A}_i|$ for all i. In this paper, we study such families in the context of certain symmetric Toeplitz matrices over a field $\mathbb{F}$. In particular, we show that if each matrix has bandwidth b and diagonal entries chosen from a finite set $\mathcal{S} \subset \mathbb{F}$, then the resulting family admits a doubling constant that depends only on b and the additive properties of $\mathcal{S}$, but is independent of the matrix dimension. Also, if the diagonals lie in the image of a fixed-dimensional linear map $L : \mathbb{F}^m \to \mathbb{F}^{b+1}$, then the doubling constant depends on m rather than b. We include examples to illustrate how one-dimensional constraints on $\mathcal{S}$ lead to especially small doubling constants.

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1. Introduction

Additive combinatorics is devoted to studying additive properties of sets in algebraic structures, typically abelian groups. A central notion is small doubling, where a finite set A has doubling constant

$$\sigma(A) = \frac{|A+A|}{|A|}.$$

We say A has small doubling if $\sigma(A)$ is not too large compared to $|A|$. Understanding sets with small doubling has led to profound structural theorems in additive number theory, such as Freiman's Theorem

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(see [1–3]), which implies that sets of small doubling in $\mathbb{Z}$ (or in a torsion-free abelian group) are contained in a generalized arithmetic progression of bounded rank. For a more modern exposition with detailed proofs and additional context, see also [5].

In this paper, we consider families of certain symmetric Toeplitz matrices over a field $\mathbb{F}$ and investigate their doubling constants. Recall that an $n \times n$ matrix T is *Toeplitz* if $T_{i,j}$ depends only on the difference $j - i$. Imposing symmetry, $T_{i,j} = T_{j,i}$, means the matrix is determined by $\{t_0, \ldots, t_b\}$ when restricted to a bandwidth b. Specifically, all entries beyond the $\pm b$-th diagonals are zero. If each diagonal parameter t_k lies in some finite set $\mathcal{S} \subset \mathbb{F}$, the family of all such bandwidth-b Toeplitz matrices can be viewed as $\mathcal{S}^{b+1}$. One key observation is that when we add two Toeplitz matrices, their diagonal parameters add componentwise in $\mathcal{S} \times \cdots \times \mathcal{S}$. Thus, the sumset $\mathcal{A} + \mathcal{A}$ can be controlled by $\mathcal{S} + \mathcal{S}$.

A frequently used inequality in one-dimensional settings (e.g., intervals in $\mathbb{Z}$ or cyclic subgroups of $\mathbb{F}$) is

$$|S + S| \;\le\; 2\,|S| \;-\; 1.$$

However, this bound is not universal: in higher-dimensional or more complicated subsets, one might only have the trivial $|S + S| \le |S|^2$. Throughout our discussion, we will specify the dimension and structure of $\mathcal{S}$. When $\mathcal{S}$ is truly 1-dimensional in $\mathbb{F}$, the $\le 2|S| - 1$ bound yields a doubling constant

$$\sigma(\mathcal{A}) \;\le\; \left(\tfrac{2|\mathcal{S}|-1}{|\mathcal{S}|}\right)^{b+1}$$

for the family $\mathcal{A}$ of bandwidth-b Toeplitz matrices. But if $\mathcal{S}$ has no such 1D restriction, we may need to replace $2|S| - 1$ by $|S|^2$ or another valid sumset bound. In any case, these estimates remain independent of the matrix dimension n, because only the number of free diagonals $(b + 1)$ matters.

We also consider a variant where the diagonal parameters $\{t_0, \ldots, t_b\}$ are constrained to lie in the image of a fixed-dimensional linear subspace. In other words, if we have $\mathcal{X} \subset \mathbb{F}^m$ (with $m \ll b$) and a linear map $L : \mathbb{F}^m \to \mathbb{F}^{b+1}$, we only allow $\big(t_0, \ldots, t_b\big) = L(\mathbf{x})$ for some $\mathbf{x} \in \mathcal{X}$. Then the family $\mathcal{A}$ of such Toeplitz matrices can be seen as $\mathrm{Toeplitz}\big(L(\mathcal{X})\big)$. When we add two matrices, their parameters add in $\mathcal{X}$. This leads to a dimension of at most m, and any sumset bound on $\mathcal{X} + \mathcal{X}$ applies. In particular, the bandwidth b no longer drives the doubling constant; rather, it is the smaller dimension m of the subspace that matters. Such families embed naturally in rank-m matrix generalized arithmetic progressions.

Thus, we reconcile the well-known sumset bounds in additive combinatorics with the specific structure of symmetric Toeplitz matrices. Our approach clarifies exactly when the factor $\big(\frac{2|S|-1}{|S|}\big)^{b+1}$ is valid (namely, in 1-dimensional settings for $\mathcal{S}$) and when one must invoke a more general (and typically larger) bound on $|S + S|$.

2. Main results

Theorem 2.1. *Let $\mathbb{F}$ be a field, and let $\mathcal{S} \subset \mathbb{F}$ be a finite set. Consider the family $\mathcal{A}$ of all symmetric Toeplitz $n \times n$ matrices of bandwidth $\le b$ whose $(b + 1)$ diagonal parameters each lie in $\mathcal{S}$. Then:*

1. *There is a natural bijection*

$$\underbrace{\mathcal{S} \times \cdots \times \mathcal{S}}_{b+1 \text{ copies}} \longleftrightarrow \mathcal{A},$$

hence

$$|\mathcal{A}| \;=\; |\mathcal{S}|^{b+1}.$$

2. *Under this bijection, the sum of two Toeplitz matrices corresponds to the componentwise sum of their diagonal parameters. Consequently,*

$$\mathcal{A} + \mathcal{A} \;\subseteq\; \big\{ T(t_0, \ldots, t_b) \;:\; t_k \in \mathcal{S} + \mathcal{S} \big\},$$

which implies

$$|\mathcal{A}+\mathcal{A}| \;\le\; \big|\mathcal{S}+\mathcal{S}\big|^{\,b+1}.$$

3. *If $|\mathcal{S}+\mathcal{S}| \le C\,|\mathcal{S}|$ for some constant C, then*

$$\sigma(\mathcal{A}) \;\le\; C^{\,b+1}.$$

In particular, if $\mathcal{S} \subset \mathbb{F}$ lies in a 1-dimensional subgroup e.g., an interval in $\mathbb{Z}$ or a cyclic subgroup in a finite field) so that $|\mathcal{S}+\mathcal{S}| \le 2|\mathcal{S}|-1$, then

$$\sigma(\mathcal{A}) \;\le\; \left(\tfrac{2|\mathcal{S}|-1}{|\mathcal{S}|}\right)^{\,b+1}.$$

Proof. (1) Each symmetric Toeplitz matrix of bandwidth $\le b$ is determined by $(b+1)$ diagonal parameters. Concretely, if

$$T = (T_{i,j})_{1\le i,j\le n},$$

then for $1 \le |i-j| \le b$, we have $T_{i,j} = t_{|i-j|}$, and for $|i-j| > b$, we have $T_{i,j} = 0$. Imposing symmetry $T_{i,j} = T_{j,i}$ forces $t_{-k} = t_k$, so effectively one needs $\{t_0, t_1, \ldots, t_b\}$ to specify T. By hypothesis, each t_k lies in $\mathcal{S}$, a finite set. Hence there is a natural bijection

$$\underbrace{\mathcal{S}\times\cdots\times\mathcal{S}}_{b+1 \text{ copies}} \;\longleftrightarrow\; \mathcal{A},$$

giving

$$|\mathcal{A}| \;=\; |\mathcal{S}|^{\,b+1}.$$

(2) Let $T_1, T_2 \in \mathcal{A}$ be determined by diagonal parameters $(t_0^{(1)}, \ldots, t_b^{(1)})$ and $(t_0^{(2)}, \ldots, t_b^{(2)})$, respectively. Then their sum $T_1 + T_2$ is also a symmetric Toeplitz matrix with diagonal parameters

$$\big(t_0^{(1)} + t_0^{(2)},\; \ldots,\; t_b^{(1)} + t_b^{(2)}\big).$$

Each $t_k^{(1)} + t_k^{(2)}$ belongs to $\mathcal{S}+\mathcal{S}$. Hence every element of $\mathcal{A}+\mathcal{A}$ is determined by $(b+1)$ elements of $\mathcal{S}+\mathcal{S}$, giving

$$\mathcal{A}+\mathcal{A} \;\subseteq\; \Big(\mathcal{S}+\mathcal{S}\Big)\times\cdots\times\Big(\mathcal{S}+\mathcal{S}\Big) \quad (b+1 \text{ copies}).$$

Therefore,

$$|\mathcal{A}+\mathcal{A}| \;\le\; \big|\mathcal{S}+\mathcal{S}\big|^{\,b+1}.$$

(3) Suppose $|\mathcal{S}+\mathcal{S}| \le C\,|\mathcal{S}|$ for some constant C. Then

$$|\mathcal{A}+\mathcal{A}| \;\le\; \big|\mathcal{S}+\mathcal{S}\big|^{\,b+1} \;\le\; \big(C\,|\mathcal{S}|\big)^{b+1} \;=\; C^{\,b+1}\,|\mathcal{S}|^{\,b+1}.$$

Since $|\mathcal{A}| = |\mathcal{S}|^{\,b+1}$, we get

$$\sigma(\mathcal{A}) \;\le\; C^{\,b+1}.$$

In particular, if $\mathcal{S} \subset \mathbb{F}$ is contained in a 1-dimensional subgroup (*e.g.*, an interval in $\mathbb{Z}$ or a cyclic subgroup in a finite field) so that $|\mathcal{S}+\mathcal{S}| \le 2|\mathcal{S}|-1$, then one obtains the explicit inequality

$$\sigma(\mathcal{A}) \;\le\; \left(\tfrac{2|\mathcal{S}|-1}{|\mathcal{S}|}\right)^{\,b+1}.$$

□

Example 2.2. *Let $b = 1$ and let $\mathcal{S} = \{0, 1\} \subset \mathbb{F}$ (where $\mathbb{F}$ can be any field, for instance $\mathbb{R}$). In this setting, every symmetric Toeplitz matrix of bandwidth ≤ 1 is determined by the two diagonal parameters (t_0, t_1) with $t_0, t_1 \in \{0, 1\}$. Therefore, there are*

$$|\mathcal{A}| = |\mathcal{S}|^{1+1} = 2^2 = 4$$

such matrices. Concretely, these matrices are:

$$T^{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad T^{(0,1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T^{(1,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T^{(1,1)} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

When adding two such matrices, the corresponding diagonal parameters add componentwise. Notice that

$$\mathcal{S} + \mathcal{S} = \{0 + 0,\ 0 + 1,\ 1 + 0,\ 1 + 1\} = \{0, 1, 2\}.$$

Hence,

$$|\mathcal{S} + \mathcal{S}| = 3.$$

Using Theorem 2.1, we obtain

$$|\mathcal{A} + \mathcal{A}| \leq |\mathcal{S} + \mathcal{S}|^{1+1} = 3^2 = 9,$$

and so the doubling ratio satisfies

$$\sigma(\mathcal{A}) \leq \frac{9}{4}.$$

Moreover, since $\mathcal{S}$ is contained in a 1-dimensional subgroup of $\mathbb{F}$ (e.g., as an interval in $\mathbb{Z}$ or a cyclic subgroup in a finite field), we can also note that

$$|\mathcal{S} + \mathcal{S}| \leq 2|\mathcal{S}| - 1 = 2 \cdot 2 - 1 = 3,$$

which confirms the same numerical bound.

Let $\mathbb{F}$ be a field, and let $b \geq 0$. Suppose we have a finite set $\mathcal{X} \subset \mathbb{F}^m$ for some integer m, and a linear map

$$L\colon \mathbb{F}^m \longrightarrow \mathbb{F}^{b+1}.$$

For each $\mathbf{x} \in \mathcal{X}$, define $\mathrm{Toeplitz}\big(L(\mathbf{x})\big)$ to be the symmetric Toeplitz matrix of bandwidth $\leq b$ whose $(b+1)$ diagonal parameters are exactly $L(\mathbf{x})$. Then set

$$\mathcal{A} = \Big\{ \mathrm{Toeplitz}\big(L(\mathbf{x})\big) \ \Big|\ \mathbf{x} \in \mathcal{X} \Big\}.$$

Theorem 2.3. *The following statements hold:*

1. *If there is a bijection $\mathbf{x} \mapsto \mathrm{Toeplitz}\big(L(\mathbf{x})\big)$ between $\mathcal{X}$ and $\mathcal{A}$, then $|\mathcal{A}| = |\mathcal{X}|$.*
2. *$|\mathcal{A} + \mathcal{A}| \leq |\mathcal{X} + \mathcal{X}|$. Moreover, if $|\mathcal{X} + \mathcal{X}| \leq K\,|\mathcal{X}|$ for some constant K, then*

$$\sigma(\mathcal{A}) \leq K.$$

3. (a) *If $\mathbb{F} = \mathbb{Q}$ (or a subfield of $\mathbb{R}$ in which all entries of $\mathcal{A}$ lie in a discrete subset), then after possibly multiplying each M_k by a common denominator to clear fractions, there exist integers $N_k > 0$ (allowing negative indices as well) such that*

$$\mathcal{A} \subseteq \Big\{ \sum_{k=1}^{m} \ell_k M_k \;\Big|\; -N_k \le \ell_k \le N_k,\ \ell_k \in \mathbb{Z} \Big\},$$

where each $M_k = \mathrm{Toeplitz}\big(L(\mathbf{e}_k)\big)$, then $\mathcal{A}$ is contained in a rank-mk integer coefficient matrix GAP.

(b) *If $\mathcal{X}$ itself can be covered by a one-dimensional arithmetic progression or by a higher-rank generalized arithmetic progression in $\mathbb{F}^m$, then $\mathcal{A}$ can be covered by the corresponding matrix progression $\{\sum_{k=1}^{m} \ell_k M_k : \ell_k \in \mathcal{P}\}$ for an appropriate set $\mathcal{P} \subset \mathbb{F}$.*

Proof. (1) By construction, $L(\mathbf{x}) \in \mathbb{F}^{b+1}$ specifies exactly $(b+1)$ diagonal parameters of a bandwidth-b symmetric Toeplitz matrix. Different $\mathbf{x} \in \mathbb{F}^m$ give distinct diagonal-parameter tuples (because L is a linear map and $\mathbf{x} \mapsto L(\mathbf{x})$ is injective on $\mathcal{X}$ if we regard $\mathcal{X}$ as a set). Hence there is a bijection between $\mathcal{X}$ and $\mathcal{A}$, giving

$$|\mathcal{A}| = |\mathcal{X}|.$$

(2) If $T_1 = \mathrm{Toeplitz}\big(L(\mathbf{x})\big)$ and $T_2 = \mathrm{Toeplitz}\big(L(\mathbf{y})\big)$, then their sum is

$$T_1 + T_2 \;=\; \mathrm{Toeplitz}\big(L(\mathbf{x}) + L(\mathbf{y})\big) \;=\; \mathrm{Toeplitz}\big(L(\mathbf{x} + \mathbf{y})\big),$$

using linearity of L. Thus addition in $\mathcal{A}$ corresponds to addition in $\mathcal{X}$. Therefore, every element of $\mathcal{A}+\mathcal{A}$ arises from some $\mathbf{x} + \mathbf{y} \in \mathcal{X} + \mathcal{X}$. Consequently,

$$|\mathcal{A} + \mathcal{A}| \;\le\; |\mathcal{X} + \mathcal{X}|.$$

If in addition we have a doubling condition $|\mathcal{X} + \mathcal{X}| \le K\,|\mathcal{X}|$, it follows that

$$\sigma(\mathcal{A}) \;\le\; \frac{K\,|\mathcal{X}|}{|\mathcal{X}|} \;=\; K.$$

Hence the doubling constant for $\mathcal{A}$ depends on m and the sumset properties of $\mathcal{X}$, *not* on b or n.

(3) Define

$$M_k := \mathrm{Toeplitz}\big(L(\mathbf{e}_k)\big),$$

so any $T(\mathbf{x}) = \mathrm{Toeplitz}\big(L(\mathbf{x})\big)$ can be formally written as $\sum_{k=1}^{m} x_k M_k$.

(a)

If $\mathbb{F} = \mathbb{Q}$ and all entries of each M_k lie in $\mathbb{Q}$, we can multiply each M_k by a suitable integer to clear denominators in all matrix entries. Hence we may assume each M_k has *integer* entries, and likewise each x_k in $\mathcal{X}$ is an integer (or rational) lying in a bounded set. It follows that each $\mathbf{x} \in \mathcal{X}$ yields a finite range of integer linear combinations $\sum_{k=1}^{m} \ell_k M_k$. By allowing $-N_k \le \ell_k \le N_k$ for some integer N_k (N_k can be at least as large as the maximum possible absolute value of x_k), we cover all matrices in $\mathcal{A}$. Thus

$$\mathcal{A} \subseteq \Big\{ \sum_{k=1}^{m} \ell_k M_k \;:\; -N_k \le \ell_k \le N_k,\ \ell_k \in \mathbb{Z} \Big\},$$

which is precisely a rank-mk integer matrix GAP.

(b)

Even if $\mathbb{F}$ is not discrete (e.g. $\mathbb{R}$), one can still say: if $\mathcal{X}$ is contained in a generalized arithmetic progression in $\mathbb{F}^m$, then $\mathcal{A}$ is contained in the corresponding matrix progression. Indeed, covering $\mathcal{X} \subseteq \mathcal{P}$ by a set $\mathcal{P} = \{\sum_{k=1}^{m} \ell_k u_k : \ell_k \in I_k\} \subset \mathbb{F}^m$ implies every $\mathbf{x} \in \mathcal{X}$ has coordinates $\mathbf{x} = \sum_{k=1}^{m} \ell_k u_k$. Hence

$$T(\mathbf{x}) = \sum_{k=1}^{m} \ell_k \operatorname{Toeplitz}\big(L(u_k)\big),$$

and the ℓ_k range over the index sets I_k. This is a rank-mk matrix GAP in the sense that the sum of two such matrices corresponds to the sum of the respective $\mathbf{x}$-coordinates. The crucial point is that the linear map L preserves sums, so the embedding follows from the same argument as in part (2) of the theorem.

□

Example 2.4. *Let $\mathbb{F} = \mathbb{Q}$ and $b = 2$. Define a linear map*

$$L : \mathbb{Q}^2 \longrightarrow \mathbb{Q}^3, \quad L(\alpha, \beta) = \big(\alpha,\ \alpha+\beta,\ \beta\big).$$

Take a finite set

$$\mathcal{X} = \Big\{ (\alpha, \beta) \ \Big|\ \alpha, \beta \in \{0, \tfrac{1}{2}, 1\} \Big\} \subset \mathbb{Q}^2.$$

This set $\mathcal{X}$ has $|\mathcal{X}| = 3 \times 3 = 9$ elements.

For each $(\alpha, \beta) \in \mathcal{X}$, let

$$T(\alpha, \beta) = \operatorname{Toeplitz}\big(L(\alpha, \beta)\big) = \operatorname{Toeplitz}\big(\alpha,\ \alpha+\beta,\ \beta\big).$$

All such 3×3 diagonal-parameter vectors $(t_0, t_1, t_2) = (\alpha, \alpha+\beta, \beta)$ determine bandwidth-2 symmetric Toeplitz matrices in the usual way:

$$T_{i,j}(\alpha, \beta) = \begin{cases} \alpha, & |i-j| = 0, \\ \alpha+\beta, & |i-j| = 1, \\ \beta, & |i-j| = 2, \\ 0, & \textit{otherwise}. \end{cases}$$

Let

$$\mathcal{A} = \big\{ T(\alpha, \beta) : (\alpha, \beta) \in \mathcal{X} \big\}.$$

Define

$$M_1 = \operatorname{Toeplitz}\big(L(1,0)\big), \quad M_2 = \operatorname{Toeplitz}\big(L(0,1)\big).$$

Then for any $(\alpha, \beta) \in \mathcal{X}$,

$$T(\alpha, \beta) = \alpha M_1 + \beta M_2,$$

which shows the addition-preserving property from Theorem 2.3(2).

By inspection, each M_k has rational entries (since $\alpha, \beta \in \{\frac{1}{2}, 1\}$ are rational). We can clear denominators by multiplying each M_k by 2, obtaining integer-valued matrices $\widetilde{M_k} = 2 M_k$. Moreover, each $(\alpha, \beta) \in \mathcal{X}$ satisfies $\alpha, \beta \in \{0, \frac{1}{2}, 1\}$, so

$$T(\alpha, \beta) = \alpha M_1 + \beta M_2 = \tfrac{1}{2}\big(2\alpha \widetilde{M_1} + 2\beta \widetilde{M_2}\big).$$

Since $2\alpha, 2\beta \in \{0, 1, 2\}$, these coefficients are integers in the range $[-2, 2]$. Hence

$$\mathcal{A} \subseteq \left\{ \sum_{k=1}^{2} \ell_k \widetilde{M}_k \;\middle|\; -2 \le \ell_k \le 2,\ \ell_k \in \mathbb{Z} \right\},$$

which is precisely a rank-2 integer-coefficient matrix generalized arithmetic progression (GAP). This shows item (3)(a) of Theorem 2.3.

Because addition in $\mathcal{A}$ corresponds to addition in $\mathcal{X}$, we also have $|\mathcal{A}+\mathcal{A}| \le |\mathcal{X}+\mathcal{X}|$, so the doubling constant $\sigma(\mathcal{A})$ depends on the dimension of $\mathcal{X}$ (i.e. $m = 2$ here) and not on the bandwidth $b = 2$. In particular, if $\mathcal{X}$ itself has a small doubling ratio in $\mathbb{Q}^2$, then so does the Toeplitz family $\mathcal{A}$.

The following result is analogous to Freiman's inverse theorem for subsets of abelian groups and indicates that small doubling forces a strong arithmetic structure even in the setting of structured matrices.

Theorem 2.5. *Let $\mathcal{A}$ be a family of symmetric Toeplitz $n \times n$ matrices of bandwidth $\le b$ whose diagonal parameters are chosen from a finite set $\mathcal{S} \subset \mathbb{F}$. Suppose that*

$$\sigma(\mathcal{A}) = \frac{|\mathcal{A}+\mathcal{A}|}{|\mathcal{A}|} \le K,$$

for some constant K. Then there exist constants $r = r(K)$ and $\delta = \delta(K) > 0$ such that

$$\mathcal{A} \subset P,$$

where P is a matrix generalized arithmetic progression (GAP) of rank at most r satisfying

$$|P| \le f(K)\,|\mathcal{A}|,$$

and, in addition, there exists a subset $\mathcal{A}' \subset \mathcal{A}$ with

$$|\mathcal{A}'| \ge \delta\,|\mathcal{A}|$$

such that $\mathcal{A}'$ is contained in a GAP of rank at most r.

Proof. Define

$$P = \{(t_0, t_1, \ldots, t_b) \in \mathcal{S}^{b+1} : T = \mathrm{Toeplitz}(t_0, t_1, \ldots, t_b) \in \mathcal{A}\}.$$

By the construction of $\mathcal{A}$, there is a natural bijection

$$\Phi\colon \mathcal{A} \to P, \quad \Phi(T) = (t_0, t_1, \ldots, t_b).$$

Since the addition of two symmetric Toeplitz matrices corresponds to the componentwise addition of their diagonal parameters, we have for any $T_1, T_2 \in \mathcal{A}$

$$\Phi(T_1 + T_2) = \Phi(T_1) + \Phi(T_2),$$

where the sum on the right is the usual sum in $\mathbb{F}^{b+1}$. In particular,

$$|\mathcal{A}+\mathcal{A}| = |\Phi(\mathcal{A}+\mathcal{A})| = |P+P|.$$

By hypothesis,

$$|\mathcal{A}+\mathcal{A}| \le K\,|\mathcal{A}|,$$

and since $|\mathcal{A}| = |P|$, we deduce that

$$|P+P| \le K\,|P|.$$

Thus, the set $P \subset \mathbb{F}^{b+1}$ has small doubling in the ambient abelian group.

Now, by Freiman's inverse theorem (see [2] for a precise statement in torsion-free abelian groups), there exist constants $r = r(K)$ and $\delta = \delta(K) > 0$, and a generalized arithmetic progression (GAP) $Q \subset \mathbb{F}^{b+1}$ of rank at most r such that:

$$P' \subset Q \quad \text{and} \quad |P'| \geq \delta\, |P|,$$

and moreover,

$$|Q| \leq f(K)\, |P|,$$

where f is a function depending only on K.

Define the corresponding matrix GAP by

$$\widetilde{Q} \;=\; \{\text{Toeplitz}(t_0, t_1, \ldots, t_b) : (t_0, t_1, \ldots, t_b) \in Q\}.$$

Since Φ is a bijection, if we set

$$\mathcal{A}' \;=\; \Phi^{-1}(P'),$$

then $\mathcal{A}' \subset \mathcal{A}$ and

$$|\mathcal{A}'| = |P'| \geq \delta\, |P| = \delta\, |\mathcal{A}|.$$

Moreover, by the definition of $\widetilde{Q}$, we have

$$\mathcal{A}' \subset \widetilde{Q} \quad \text{and} \quad |\widetilde{Q}| = |Q| \leq f(K)\, |\mathcal{A}|.$$

Thus, a large portion of $\mathcal{A}$ (at least a δ-fraction) is contained in a matrix GAP of rank at most r and size at most $f(K)\, |\mathcal{A}|$.

□

Remark 2.6. *Theorem 2.5 states that if the family $\mathcal{A}$ has small doubling, then a large portion of it is structured—namely, it is contained in a matrix GAP of uniformly bounded rank. In particular, when the diagonal entries are drawn from a 1-dimensional subset (so that the optimal doubling bound $\left(\frac{2|\mathcal{S}|-1}{|\mathcal{S}|}\right)^{b+1}$ holds), the structure is especially rigid.*

3. Conclusion

In this paper, we have studied the small doubling properties for families of symmetric Toeplitz matrices. Our results show that when the diagonal entries of these matrices are chosen from a finite set $\mathcal{S}$, the resulting doubling constant depends only on the number of free parameters, i.e. the bandwidth b, and the additive structure of $\mathcal{S}$, and is independent of the overall matrix dimension. Moreover, by imposing additional structure via a linear map $L : \mathbb{F}^m \to \mathbb{F}^{b+1}$, we demonstrated that the effective doubling constant can be reduced further to depend solely on the lower dimension m of the parameter space. Our examples illustrate that when $\mathcal{S}$ is contained in a 1-dimensional subgroup (or an arithmetic progression), one obtains especially tight doubling bounds.

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