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EAQEC codes from polycyclic codes over the ring R_l

Research Article

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Abstract: In this paper, we study polycyclic codes over the ring $R_l = \frac{\mathbb{F}_q[w]}{\langle w^l - 1\rangle}$ with $q = p^k$ where k is a positive integer and p is an odd prime. We explore LCD annihilator, self-dual, self-orthogonal codes over R_l and polycyclic codes over R_l. Moreover, we provide a structure of entanglement-assisted quantum error-correcting codes based on the developed polycyclic codes through the inclusion of dual contained conditions. Subsequently, some LCD hull code-derived entanglement-assisted quantum error-correcting code examples are presented.

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1. Introduction

Coding Theory is a discipline encompassing mathematics and computer science that emphasizes the development of error-detecting and error-correcting codes for improving data and communication reliability. Cyclic codes have played a significant role in communication and data storage. In the 1950s [22], Prange developed cyclic codes. If a code is linear and each cyclic shift of the coordinates of a codeword leads to another codeword, then the code is cyclic. Many researchers have investigated cyclic codes over several finite rings [6, 30, 31]. Constacyclic codes [5] are an exceptional extension of cyclic codes. A linear

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code C of length n is said to be λ-constacyclic if $(c_0, c_1, \ldots, c_{n-1}) \in C$ implies $(\lambda c_{n-1}, c_0, \ldots, c_{n-2}) \in C$. For constacyclic codes over finite fields and finite rings, one can refer to [9–12],[17, 18].

Another notable enhancement of cyclic codes is polycyclic codes. That is, codes that can be considered as ideals of a factor ring $\frac{\mathbb{F}_q[x]}{\langle f(x)\rangle}$. When $f(x) = x^n - \alpha$ where $\alpha \neq 0 \in \mathbb{F}_q$, then polycyclic codes transform into constacyclic codes. When $\alpha = 1$, then polycyclic code is nothing but cyclic codes. A linear code is polycyclic if and only if its Euclidean dual code is sequential which is not always polycyclic, according to a study accomplished in 2009 by Lopez-Permouth et al. [19]. In 2016, Alahmadi et al. [2] introduced the annihilator dual codes over $\mathbb{F}_q$ and showed that the annihilator dual codes of polycyclic codes over $\mathbb{F}_q$ are also polycyclic. In 2022, Wei Qi studied the polycyclic codes over $\mathbb{F}_q + u\mathbb{F}_q$ with $u^2 = u$. Later, in 2024 [24], the properties of idempotent generator of cyclic codes were extended to polycyclic codes over the finite field $\mathbb{F}_q$. In [16], the authors studied the structure of polycyclic codes over the ring $R = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q; u^2 = \alpha u, v^2 = v$ and $uv = vu = 0$, where α is a unit element in R. Recently, in [4], the authors investigated quasi-polycyclic (QPC) codes and skew quasi-polycyclic (SQPC) codes. Further, they analyzed the role of polycyclic codes in quantum error-correcting codes (QECCs). When compared to traditional QECCs codes, entanglement-assisted quantum error-correcting codes (EAQECCs) enhance error correction effectiveness and optimize the utilization of resources. By leveraging pre-shared entanglement between the sender and receiver, these codes achieve strong error correction performance while requiring fewer physical qubits. Introduced by Brun et al. in 2006 [7], EAQECCs integrate entanglement into the framework of quantum error correction, simplifying the error correction process and increasing the capacity for transmitting and receiving information in quantum communication systems.

Inspired by the above-mentioned works and recent developments, we focused to construct the polycyclic codes over $\mathbb{F}_q$ using the ring $R_l = \frac{\mathbb{F}_q[w]}{\langle w^l - 1\rangle}$. Moreover, we discussed the construction of entanglement assistant quantum error correcting code using the constructed polycyclic codes by establishing dual containing conditions. The proposed construction provides greater flexibility in designing quantum codes and enhances their error-correcting capability. Moreover, the new EAQECCs derived from our approach demonstrate improved parameters, and their performance is shown to surpass several existing codes through detailed comparison.

The paper is organized as follows. Section 2 introduces fundamental concepts and presents the decomposition of linear codes over R_l. Section 3 examines the structure of polycyclic codes over the ring R_l and defines the Gray map from R_l to $\mathbb{F}_q^l$. In Section 4, we construct entanglement-assisted quantum error-correcting codes (EAQECCs) from LCD hull codes, accompanied by illustrative examples. Finally, in Section 5, we have summarized our key findings.

2. Preliminaries

Let $\mathbb{F}_q$ be the finite field of order $q = p^m$ with characteristic p where p is a prime and $m \geq 1$. Now from [3] consider the ring $R_l = \frac{\mathbb{F}_q[w]}{\langle w^l - 1\rangle}$ where l divides $(p-1)$. The elements of the ring R_l are in the form $a_0 + wa_1 + w^2a_2 + \cdots + w^{l-1}a_{l-1}$.

Note that

$$w^l - 1 = \prod_{i=1}^{l}(w - w_i),\ w_i \in \mathbb{F}_q$$

Consider $G_i = w - w_i$ and $\overline{G_i} = \frac{w^l-1}{G_i}$. It is clear that $\gcd(G_i, \overline{(G_i)}) = 1$, then there exist $a_i, b_i \in \mathbb{F}_q[w]$ such that $a_iG_i + b_i\overline{G_i} = 1$. Choose $e_i = b_i\overline{G_i}$ for all $i = 1, 2, 3, \cdots, l$. Then we have $e_ie_j = 0$ when $i \neq j$, $e_i^2 = e_i$ and $\sum_{i=1}^{l} e_i = 1$.

By Chinese Remainder Theorem,

$$\bigoplus_{i=1}^{l} e_i R_l \cong \bigoplus_{i=1}^{l} e_i \mathbb{F}_q.$$

Hence any element $r \in R_l$ can be written as $r = \sum_{i=1}^{l} e_i r_i$ where $r_i \in \mathbb{F}_q$.
If $C \subseteq R_l^n$ is a linear code then define $C_i \subseteq \mathbb{F}_q^n$ as
$C_i = \{r_i \in \mathbb{F}_q^n \mid \exists\ r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_l \in \mathbb{F}_q^n$ such that $\sum_{i=1}^{l} e_i r_i \in C\}$. Then C_i is a linear code of length n over $\mathbb{F}_q$ for $1 \leq i \leq l$. Hence, C can be expressed as $C = \bigoplus_{i=1}^{l} e_i C_i$.

Definition 2.1. *Let C be a linear code of length n over R_l. Let $a = (a_0, a_1, \ldots a_{n-1}) \in R_l^n$ where a_0 is a unit element of R_l. The a-polycyclic shift $\rho_a : R_l^n \to R_l^n$ is defined by*

$$\rho_a(c_0, c_1, \ldots c_{n-1}) = (0, c_0, c_1, \ldots c_{n-2}) + c_{n-1}(a_0, a_1, a_2, \ldots a_{n-1})$$

for all $c = (c_0, c_1, \ldots, c_{n-1}) \in R_l^n$. The a-sequential shift $\tau_a : R_l^n \to R_l^n$ is defined by

$$\tau_a(c_0, c_1, \ldots c_{n-1}) = (c_1, c_2, \ldots c_{n-1}, c_0 a_0 + c_1 a_1 + \cdots + c_{n-1} a_{n-1})$$

for all $c = (c_0, c_1, \ldots, c_{n-1}) \in R_l^n$.
A linear code $C \subseteq R_l^n$ is called a-polycyclic code if $\rho_a(C) \subseteq C$. A linear code $C \subseteq R_l^n$ is called a-sequential code if $\tau_a(C) \subseteq C$.

Polynomials can be deployed to identify the elements of R_l^n. For instance,
$c = (c_0, c_1, \ldots, c_{n-1}) \in R_l^n$ considering $c(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} \in R_l[x]/\langle x^n - a(x)\rangle$. Consequently, R_l^n and $R_l[x]/\langle x^n - a(x)\rangle$ corresponds one to one. If and only if its polynomial representation provides an ideal within the ring $R^a = R_l[x]/\langle x^n - a(x)\rangle$, then a linear code C is a polycyclic code of length n over R_l.

Theorem 2.2. *Let C be an a-polycyclic code over the ring R_l, then the corresponding image set ϕ is an $R_l[x]$-module over $R_l[x]/\langle x^n - a(x)\rangle$.*

Definition 2.3. *As in Definition 3.1 of [23],let C be an a-polycyclic code of length n. Let $\delta(x), \zeta(x) \in R^a$. Then the annihilator product of $\delta(x)$ and $\zeta(x)$ is defined as*

$$\langle \delta(x), \zeta(x) \rangle_a = s(0)$$

where $\delta(x)\zeta(x) \equiv s(x) (mod\ x^n - a(x))$ and $deg(s(x)) \leq n - 1$. The annihilator dual code C^o of C is defined to be

$$C^o = \{\zeta(x) \in R^a \mid \langle \delta(x), \zeta(x) \rangle_a = s(0) = 0 \text{ for all } \delta(x) \in C\}.$$

The code C is called an annihilator self-orthogonal code (resp., annihilator self-dual code, annihilator LCD code) provided that $C \subseteq C^o$ (resp., $C = C^o, C \cap C^o = \{0\}$). The annihilator of C is

$$Ann(C) = \{\zeta(x) \in R^a \mid \delta(x)\zeta(x) = 0 \in R^a \text{ for all } \delta(x) \in C\}.$$

The following are some important results that we can use.

Theorem 2.4. *[2] Let C be an a-polycyclic code of length n over $\mathbb{F}_q$. Let $g(x)$ be the generator polynomial and $h(x) = \frac{x^n - a(x)}{g(x)}$ be the check polynomial of C, then $C^o = \langle h(x) \rangle$.*

Lemma 2.5. *[23] Let $a = (a_0, a_1, \cdots, a_{n-1}) \in \mathbb{F}_q^n$ with $a_0 \neq 0$, C be an a-polycyclic code of length n over $\mathbb{F}_q$, then $\delta(x)\zeta(x)$ is non-degenerate, and thus $C^o = Ann(C)$.*

3. Codes over the ring R_l

Define a Gray map from R_l to $\mathbb{F}_q^l$ by $\phi(e_1r_1 + e_2r_2 + \cdots + e_lr_l) = (r_1, r_2, \cdots, r_l)$. One can check that the defined map is an isomorphism between the two rings and it can be extended to n copies. If $wt_G(r) = wt_H(\phi(r))$ then it preserves distance where $r \in R_l$. In this section we have studied the structure of Gray map and its properties.

Theorem 3.1. *Let $C = \bigoplus_{i=1}^{l} e_iC_i$ be an a-polycyclic code over R_l iff $C_i's$ are a-polycyclic codes over $\mathbb{F}_q^l$.*

Proof. Since C is a-polycyclic code over R_l we have $c = (c_0, c_1, \cdots, c_{n-1}) \in C$ implies $\rho_a(c) = (0, a_1, a_2, \cdots, a_{n-2}) + c_{n-1}(a_0, a_1, \cdots, a_{n-1})$. Hence, we have $\rho_a(c) = e_1\rho_{a_1}(c^1) + e_1\rho_{a_2}(c^2) + \cdots + e_1\rho_{a_l}(c^l)$ where $c_j = \sum_{i=1}^{l} c_j^i e_i$, $c_j^i \in C_i \subseteq \mathbb{F}_q^n$ implies $\rho_{a_i}(c^i) \in C_i$. Thus, $C_i's$ are a-polycyclic codes over $\mathbb{F}_q^l$. Conversely, we assume that $C_i^{'}$ be the polycyclic codes over $\mathbb{F}_q$ for $i = 0, 1, 2, \ldots, l$. We have $c = (c_0, c_1, \cdots, c_{n-1}) \in C$, where $c_j = \sum_{i=1}^{l} c_j^i e_i$, $c_j^i \in C_i \subseteq \mathbb{F}_q^n$ implies $\rho_{a_i}(c^i) \in C_i$ for $i = 1, 2, \ldots, l$. Now $\rho_a(c) = \sum_{i=1}^{l} \eta_i\rho_a(c^i) \in \bigoplus_{i=1}^{l} \eta_iC_i = C$. Thus, C is an a-polycyclic code of length n over R_l. □

Theorem 3.2. *Let $C = \bigoplus_{i=1}^{l} e_iC_i$ be an a-polycyclic code over R_l iff $C = \langle e_1g_1(x), e_2g_2(x), \cdots, e_lg_l(x)\rangle$, where $C_i = \langle g_i(x)\rangle$ in $\mathbb{F}_q$.*

Proof. Let $C = \bigoplus_{i=1}^{l} e_iC_i$ be an a-polycyclic code over R_l. Let $c(x) \in C = \bigoplus_{i=1}^{l} e_iC_i$, then there exists $p_i(x) \in R^a$ such that

$$\sum_{i=1}^{l} e_ip_i(x)g_i(x) = c(x)$$

$$\left(\sum_{i=1}^{l} e_ip_i(x)\right)\left(\sum_{i=1}^{l} e_ig_i(x)\right) = c(x)$$

Then $c(x) \in \langle g(x)\rangle$, $\langle g(x)\rangle \subseteq \bigoplus_{i=1}^{l} e_iC_i$. Since $C = \bigoplus_{i=1}^{l} e_iC_i$ be a a-polycyclic code over R_l, then by Theorem 3.1, each C_i is a-polycyclic code over $\mathbb{F}_q$. Hence, by Theorem 2.4, we have $C_i = \langle g_i(x)\rangle$ and $g_i(x)|x^n - a_i(x)$. Then there exists $h_i(x) \in R^a$ such that $g_i(x)h_i(x) = x^n - a_i(x)$. Therefore $e_ig_i(x)h_i(x) = e_i(x^n - a_i(x))$ and hence

$$\sum_{i=1}^{l} e_ig_i(x)h_i(x) = (x^n - a_i(x))$$

$$\left(\sum_{i=1}^{l} e_ig_i(x)\right)\left(\sum_{i=1}^{l} e_ih_i(x)\right) = (x^n - a_i(x))$$

Thus, we have $C = \langle\sum_{i=1}^{l} e_ig_i(x)h_i(x)\rangle$. □

Definition 3.3. *Let C be an a linear code of length $n = lm$ then C is a $(a_1, a_2, \cdots, a_l)$-quasi polycyclic code (QPC) if for any $c_0, c_1, \cdots, c_l \in C$ we have $\rho_{a_0}(c_0), \rho_{a_1}(c_1), \cdots, \rho_{a_l}(c_l) \in C$ where $c_i \in \mathbb{F}_q^m$ and ρ_{a_i} is a polycyclic shift operator defined in Definition 2.1.*

Theorem 3.4. *Let $C = \bigoplus_{i=1}^{l} e_iC_i$ be an a-polycyclic code over R_l iff $\phi(C)$ is a $(a_1, a_2, \cdots, a_l)$-Quasi Polycyclic code over $\mathbb{F}_q$.*

Proof. Let C be an a-polycyclic code of length lm. Then, for every codeword $c \in C$, C satisfies the polycyclic shift operator. Let $c = (c_0, c_1, \ldots, c_{n-1}) \in R_l^n$ and $a = (a_0, a_1, \ldots a_{n-1}) \in R_l^n$ where

$c_j = \sum_{i=1}^{l} a_j^i \eta_i$ for $0 \leq j \leq n-1$. We assume that, if C is a-polycyclic code implies C is a quasi-polycyclic code.

$$\begin{aligned}\rho_a(c) &= (0, c_0, c_1, \ldots, c_{n-2}) + c_{n-1}(a_0, a_1, a_2, \ldots, a_{n-1}) \\ &= \sum_{i=1}^{l} \lambda_i[(0, c_0^i, c_1^i, \ldots, c_{n-2}^i) + c_{n-1}^i(a_0^i, a_1^i, a_2^i, \ldots, a_{n-1}^i)].\end{aligned}$$

$$\begin{aligned}\phi(\rho_a(c)) &= \phi((0, c_0, c_1, \ldots, c_{n-2}) + c_{n-1}(a_0, a_1, a_2, \ldots, a_{n-1})) \\ &= (0, c_0^0, c_1^0, \ldots, c_{n-2}^0) + c_{n-1}^0(a_0^0, a_1^0, a_2^0, \ldots, a_{n-1}^0), (0, c_0^1, c_1^1, \ldots, c_{n-2}^1) + \\ &\quad c_{n-1}^1(a_0^1, a_1^1, a_2^1, \ldots, a_{n-1}^1), \ldots\ldots, (0, c_0^l, c_1^l, \ldots, c_{n-2}^l) + c_{n-1}^l(a_0^l, a_1^l, a_2^l, \ldots, a_{n-1}^l).\end{aligned}$$

The converse part is true. That is, if C is a quasi-polycyclic code then C is a-polycyclic code.

Hence proved. □

Theorem 3.5. *Let $\alpha(x), \beta(x) \in R^a$. Then $\langle \alpha(x), \beta(x) \rangle_a$ is a non-degenerate symmetric R_l-bilinear form.*

Proof. For any $\alpha, \beta, \gamma \in R^n$, $k \in R$, $\langle k(\alpha+\beta), \gamma \rangle_a = r(0)$, where

$$\begin{aligned}[k(\alpha+\beta)\gamma](x) &\equiv r(x)(mod\ x^n - a(x)) \\ k[\alpha(x)\gamma(x)] + k[\beta(x)\gamma(x)] &\equiv r(x)(mod\ x^n - a(x))\end{aligned}$$

on the other hand,

$$\begin{aligned}\langle k\alpha(x), \gamma(x) \rangle_a &= r_1(0) \text{ where } k[\alpha(x)\gamma(x)] \equiv r_1(0)(mod\ x^n - a(x)) \\ \langle k\beta(x), \gamma(x) \rangle_a &= r_2(x) \text{ where } k[\beta(x)\gamma(x)] \equiv r_2(x)(mod\ x^n - a(x)),\end{aligned}$$

using the property compatibility with addition, we have $r(x) = r_1(x) + r_2(x)$. Thus, $\langle k(\alpha+\beta), \gamma \rangle_a = k\langle \alpha, \gamma \rangle_a + k\langle \beta, \gamma \rangle_a$ is bilinear. Since the ring R is commutative, we have $\langle \beta, \gamma \rangle_a = \langle \gamma, \beta \rangle_a$. In order to prove $\langle ., . \rangle_a$ is non-degenerate, it is enough to prove that the Radicals of R is $\{0\}$. Suppose not, then there exists $\beta \neq 0 \in R(R^n)$ such that $\langle \alpha, \beta \rangle_a = 0$ for all $\alpha \in R$. Since $\alpha, \beta \in R^n$, it can be uniquely represented by $\alpha = e_1\alpha_1 + e_2\alpha_2 + \cdots + e_l\alpha_l$, $\beta = e_1\beta_1 + e_2\beta_2 + \cdots + e_l\beta_l$. Therefore, by using the bilinear property, we have $\langle \alpha, \beta \rangle_a = 0$. That is,

$$\langle \alpha, \beta \rangle_a = \sum_{i=1}^{l} e_i \langle \alpha_i, \beta_i \rangle_a = 0,$$

which is a contradiction Thus $\langle ., . \rangle_a$ is a non-degenerate symmetric R-bilinear form. □

Theorem 3.6. *Let C be an a-polycyclic code over R_l, we denote $CA = \{cA | c \in C\}$. Set $\epsilon_1 = (1, 0, \cdots, 0), \epsilon_2 = (0, 1, \cdots, 0), \cdots, \epsilon_n = (0, 0, \cdots, 1)$ and $A = (\langle \epsilon_i, \epsilon_j \rangle_a), 1 \leq i, j \leq n$. Then $C^o = (CA)^{\perp}$. Consequently, $(C^o)^o = C$.*

Proof. Note that $\langle u, v \rangle_a = uAv^t = \langle u, Av \rangle_a$. Thus $C^o = (CA)^{\perp}$. Using the inequality, $C^o = (CA)^{\perp}$. Since A is invertible, it follows that $(C^o)^o = (C^oA)^{\perp} = (C^o)^{\perp}A^{-1} = ((CA)^{\perp})^{\perp}A^{-1} = C$. □

Theorem 3.7. *Let C be a linear code over R_l. Then C is a-polycyclic if and only if C^o is a-polycyclic.*

Proof. Since C is a-polycyclic code over R_l, then by Theorem 3.1, every C_i is a-polycyclic codes over $\mathbb{F}_q$. Then, by [[2], Proposition 3], we have C_i^o as polycyclic code over $\mathbb{F}_q$ and again by Theorem 3.1, it is obvious that C^o is a-polycyclic codes. □

Theorem 3.8. *Let $C = \bigoplus_{i=1}^{l} e_i C_i$ be a linear code of length n over R_l. Then $C^o = \bigoplus_{i=1}^{l} e_i C_i^o$. Moreover, C is self-dual code if and only if $C_i's$ $(i = 1, 2, \ldots, l)$ are self-dual codes over $\mathbb{F}_q$.*

Proof. Let $\overline{C}_i = \{r_i \in \mathbb{F}_q^n \mid \text{there exists } r_1, r_2, \ldots, r_{i-1}, r_{i+1}, \ldots, r_l \in \mathbb{F}_q^n \text{ such that } \sum_{i=1}^{l} e_i r_i \in C^o\}$. Then $C^o = \bigoplus_{i=1}^{l} e_i \overline{C}_i$. Hence, $\overline{C}_1 \subseteq C_1^o$. If $z(x) \in C_1^o$, then $z(x) \cdot r_1(x) = 0$ for all $r_1(x) \in C_1$. Let $s = \sum_{i=1}^{l} e_i r_i \in C$. Then $e_1 z(x) s = e_1 r_1(x) z(x) = 0$, which implies $e_1 z(x) \in C^o$. According to the definition of C^o, we have $z(x) \in \overline{C}_1$. Therefore, $C_1^o \subseteq \overline{C}_1$. Hence, $\overline{C}_1 = C_1^o$, $C_i^o = \overline{C}_i$ for $i = 2, \ldots, l$. Consequently, $C^o = \bigoplus_{i=1}^{l} e_i C_i^o$.

Moreover, let C be a self-dual linear code. Then $C = C^o, i.e., \bigoplus_{i=1}^{l} e_i C_i = \bigoplus_{i=1}^{l} e_i C_i^o$, Hence $C_i^o = C_i$ for $1 \leq i \leq l$. Conversely, let $C_i's$ $(i = 1, 2 \ldots, l)$ be self-dual linear codes. Then $C_i^o = C_i$ for $1 \leq i \leq l$. Thus, $C^o = \bigoplus_{i=1}^{l} e_i C_i^o = \bigoplus_{i=1}^{l} e_i C_i = C$. Hence C is a self-dual linear code over R_l. □

Let $C = (C_1, C_2, \cdots, C_l)$ be a QPC code over $\mathbb{F}_q$ of length ml and index l where $C_i \in \frac{\mathbb{F}_q[x]}{x^m - a_i(x)}$ is a polycyclic code generated by $g_i(x)$ such that $g_i(x) | x^m - a_i(x)$. Hence, a QPC code over $\mathbb{F}_q$ is generated by the set

$$C = \{f(x)(g_1(x), g_2(x), \cdots, g_l(x)) \mid C_i = \langle g_i(x) \rangle \text{ and } f(x) \in \mathbb{F}_q[x]\}$$

This leads to the following result.

Theorem 3.9. *Let C be an a-polycyclic code over R_l. Then*

$$\phi(C) = \{f(x)(g_1(x), g_2(x), \cdots, g_l(x)) \mid C_i = \langle g_i(x) \rangle \text{ and } f(x) \in \mathbb{F}_q[x]\}.$$

and

$$\phi(C^o) = \{f_h(x)(h_1(x), h_2(x) \cdots, h_l(x)) \mid h_i(x) = \frac{x^n - a_i(x)}{g_i(x)} \text{ and } f_h(x) \in \mathbb{F}_q[x]\}.$$

As we know that if $C = \langle g(x) \rangle$ is a polycyclic code over R_l then $C^o = \langle h(x) \rangle$ where $g(x) = \frac{x^n - a(x)}{h(x)}$ is a polycyclic code. Now we see the dual of the $\phi(C)$. Consider $C = \bigoplus_{i=1}^{l} C_i$ a polycyclic code of length n over R_l, then define $Hull(C) = C \cap C^o$. C and C^o are ideals over $\frac{R_l[x]}{\langle x^n - a(x) \rangle}$ implies $Hull(C)$ is an ideal which implies $Hull(C)$ is a a-polycyclic code.

Theorem 3.10. *Let C be a linear code of length n over R_l, then $Hull(C) = \sum_{i=1}^{l} e_i Hull(C_i)$.*

Proof. $Hull(C) = \bigoplus_{i=}^{l} e_i C_i \cap (\bigoplus_{i=}^{l} e_i C_i)^o$. Using Theorem 3.8, $(\bigoplus_{i=}^{l} e_i C_i)^o = \bigoplus_{i=}^{l} e_i C_i^o$ we have $Hull(C) = \bigoplus_{i=}^{l} e_i C_i \cap \bigoplus_{i=}^{l} e_i C_i^o$. Multiplying e_i on both sides and by using linearity one can check $\bigoplus_{i=1}^{l} e_i Hull(C_i) \subseteq Hull(C)$. Similarly the converse part is true, thus $\bigoplus_{i=1}^{l} e_i Hull(C_i) = Hull(C)$. □

Theorem 3.11. *Let C be an LCD code over R_l if and only if $C_i's$ are LCD codes over $\mathbb{F}_q$*

Proof. The proof is similar to the proof of theorem 3.10. □

Theorem 3.12. *Let C be an a polycyclic code over R_l then $Hull(C) = \langle e_1 g_1'(x), e_2 g_2'(x), \cdots, e_l g_l'(x) \rangle$ where $g_i'(x) = gcd(g_i(x), h_i(x))$ in $\mathbb{F}_q[x]$.*

Proof. From Theorem 3.10, we have $\bigoplus_{i=1}^{l} e_i Hull(C_i) = Hull(C)$. Let C be a polycyclic code then C^o is a polycyclic code. Since, $Hull(C)$ is a polycyclic code over R_l there exists $g_H(x)$ such that $Hull(C) = \langle g_H(x) \rangle$. By Theorem 3.2, there exists $g_{i,H} \in \mathbb{F}_q[x]$ such that $g_H(x) = e_1 g_{1,H}(x) + e_2 g_{2,H}(x) + \cdots + e_l g_{l,H}(x)$. As $Hull(C) = C \cap C^o = \gcd(g(x), h(x))$ where $h(x) = \frac{x^n - a(x)}{g(x)}$ in $\mathbb{F}_q$ we know that $Hull(C_i) = \gcd(g_i(x), h_i(x))$, assume $H_i = Hull(C_i)$.

$$e_i g_H(x) = e_i(e_1 g_{1,H}(x) + e_2 g_{2,H}(x) + \cdots + e_l g_{l,H}(x))$$
$$e_i g_H(x) = e_i g_{i,H}(x) \subseteq \langle e_i H_i \rangle$$
$$\sum_{i=1}^{l} e_i g_H(x) = g_H(x) = \sum_{i=1}^{l} e_i H_i$$

Hence, $Hull(C) = \langle e_1 g_1'(x), e_2 g_2'(x), \cdots, e_l g_l'(x) \rangle$. □

Theorem 3.13. *If C is a polycyclic code over R_l then $Hull(\phi(C))$ is also a QPC code over $\mathbb{F}_q$.*

Proof. Since C is a polycyclic code implies $\phi(C)$ is a QPC code over $\mathbb{F}_q$ and C is a polycyclic code implies $Hull(C)$ is a polycyclic code over R_l. Thus, $Hull(\phi(C))$ is a QPC code. □

4. Quantum codes

In this section, we have constructed some quantum codes over $\mathbb{F}_q$ using the duality property. The quantum error-correcting code (QECC) was discovered by Shor in 1995 [27]. This accomplishment revolutionized quantum computing. The search for improved and novel quantum codes has continued since then. In 1998, Calderbank et al. [8] described a classical cyclic codes approach in generating quantum error-correcting codes. Consequently, a great deal of work has been devoted to develop quantum codes using algebraic properties such as cyclic, quasi-cyclic, and constacyclic codes in general polycyclic codes in codes over rings. For more details, one can refer to [3, 15, 21, 25, 28].

Definition 4.1. *Let C be an EAQECCs with parameters $[[n, k, d; c]]_q$. If $c = n - k$, the code is referred to as a maximal-entanglement EAQECC (ME EAQEC code).*

Definition 4.2. *An EAQECCs with parameters $[[n, k, d; c]]_q$ is called a weakly maximum distance separable code (WMDSC) EAQECC if $2(d-1) \geq n - k + c - 2$.*

An EAQEC code is referred to as a maximum distance separable EAQEC code (MDS EAQEC code) if its parameters $[[n, k, d; c]]_q$ achieve the entanglement-assisted quantum Singleton bound, which is $2(d-1) = n - k + c$.

Theorem 4.3. *Let C be an $[n, k, d]_q$ classical linear code over $\mathbb{F}_q$. The dual code $C^{\perp}$ has parameters $[n, n-k, d^{\perp}]_q$. Then there exist $[[n, k - dim(Hull(C)), d; n - k - dim(Hull(C))]]_q$ and $[[n, n - k - dim(Hull(C)), d^{\perp}; k - dim(Hull(C))]]_q$ EAQEC codes.*

Theorem 4.4. *Let C be an $[n, k, d]_q$ linear code over the finite field R_l and C^o be its dual code with parameters $[n, n-k, d^{\perp}]_q$. Then there exist*

$$[[ln, lk - dim(Hull(\phi(C))), d; ln - lk - dim(Hull(\phi(C)))]]_q$$

and

$$[[ln, ln - lk - dim(Hull(\phi(C))), d^{\perp}; lk - dim(Hull(C))]]_q$$

EAQEC codes.

Proof. The proof of this theorem is direct from theorem 4.3 and the results from section 3. □

Note that if C is a LCD code($Hull(C) = \{0\}$) then we have quantum code with the following parameter

$$[[(l+1)n, (l+1)k, d; (l+1)n - (l+1)k]]_q$$

and

$$[[(l+1)n, (l+1)n - (l+1)k, d^{\perp}; (l+1)k]]_q$$

The examples constructed in this section is done by magma soft algebraic system.

Example 4.5. *For $q = 3$ and $l = 2$, consider $R_2 = \frac{\mathbb{F}_3[w]}{w^2-1}$. Let $f(x) = x^6 - x^4 + x^2 + x + 1 \in R_2[x]$, choose $g(x) \in R_2[x]$ by $g(x) = \langle e_1 g_1(x), e_2 g_2(x)\rangle$ where $g_i(x) = x + 2$. Clearly the code generated by C is $[6, 2, 2]$ from which $\phi(C)$ is $[12, 4, 2]$ and $C^o = [6, 1, 6]$ from which $\phi(C^o) = [12, 2, 6]$. Also, $Hull(C) = [6, 1, 6]$ being that $\phi(Hull(C)) = [12, 2, 6]$. Hence we have a EAQECC as $[[12, 6, 6, 2]]$ over $\mathbb{F}_3$.*

Example 4.6. *For $q = 3$ and $l = 2$, consider $R_2 = \frac{\mathbb{F}_3[w]}{w^2-1}$. Let $f(x) = x^7 - x^2 + x + 1 \in R_2[x]$, choose $g(x) \in R_2[x]$ by $g(x) = \langle e_1 g_1(x), e_2 g_2(x)\rangle$ where $g_i(x) = x^3 + 2x + 1$. Clearly the code generated by C is $[7, 4, 3]$ from which $\phi(C)$ has the parameter $[14, 8, 3]$ and $C^o = [7, 3, 4]$ from which $\phi(C^o) = [14, 6, 4]$. Also, $Hull(C) = \{0\}$ being that $\dim(\phi(Hull(C))) = 0$. Hence we have a EAQECC as $[[14, 6, 4, 8]]$ over and $[[14, 8, 3, 6]]$ $\mathbb{F}_3$.*

Example 4.7. *For $q = 3^2$ and $l = 2$, consider $R_2 = \frac{\mathbb{F}_{3^2}[w]}{w^2-1}$. Let $f(x) = x^{10} + 2x^9 + 2x^8 + 2x^7 + 2x^5 + 2x^4 + 2x^2 + x + 1 \in R_2[x]$, choose $g(x) \in R_2[x]$ by $g(x) = \langle e_1 g_1(x), e_2 g_2(x)\rangle$ where $g_i(x) = x^7 + 2x^6 + x^5 + 2x^4 + 2x^2 + 1$. Clearly the code generated by C is $[10, 3, 5]$ from which $\phi(C)$ has the parameter $[20, 6, 5]$ and $C^o = [10, 7, 2]$ from which $\phi(C^o) = [20, 14, 2]$. Also, $Hull(C) = [10, 1, 6]$ being that $\phi(Hull(C)) = [20, 2, 6]$. Hence we have a EAQECC as $[[20, 12, 2, 4]]$ over and $[[20, 4, 5, 12]]$ $\mathbb{F}_{3^2}$.*

Example 4.8. *For $q = 3$ and $l = 2$, consider $R_2 = \frac{\mathbb{F}_3[w]}{w^2-1}$. Let $f(x) = x^7 + x^4 - x^2 + x + 1 \in R_2[x]$, choose $g(x) \in R_2[x]$ by $g(x) = \langle e_1 g_1(x), e_2 g_2(x)\rangle$ where $g_i(x) = x^2 + x + 2$. Clearly the code generated by C is $[7, 5, 2]$ from which $\phi(C)$ has the parameter $[14, 10, 2]$ and $C^o = [7, 2, 4]$ from which $\phi(C^o) = [14, 4, 4]$. Also, $Hull(C) = \{0\}$ being that $\dim(\phi(Hull(C))) = 0$. Hence we have a EAQECC as $[[14, 4, 4, 10]]$ over and $[[14, 10, 2, 4]]$ $\mathbb{F}_3$.*

Example 4.9. *For $q = 3$ and $l = 2$, consider $R_2 = \frac{\mathbb{F}_3[w]}{w^2-1}$. Let $f(x) = x^7 + x^4 - x^2 + x + 1 \in R_2[x]$, choose $g(x) \in R_2[x]$ by $g(x) = \langle e_1 g_1(x), e_2 g_2(x)\rangle$ where $g_i(x) = x^2 + x + 2$. Clearly the code generated by C is $[7, 5, 2]$ from which $\phi(C)$ has the parameter $[14, 10, 2]$ and $C^o = [7, 2, 4]$ from which $\phi(C^o) = [14, 4, 4]$. Also, $Hull(C) = \{0\}$ being that $\dim(\phi(Hull(C))) = 0$. Hence we have a EAQECC as $[[14, 4, 4, 10]]$ over and $[[14, 10, 2, 4]]$ $\mathbb{F}_3$.*

Example 4.10. *For $q = 7^2$ and $l = 2$, consider $R_2 = \frac{\mathbb{F}_{7^2}[w]}{w^2-1}$. Let $f(x) = \alpha^2 x^{10} + \alpha^{18} x^9 + \alpha^7 x^8 + \alpha^{21} x^7 + \alpha^{12} x^6 + \alpha^{39} x^5 + \alpha^{21} x^4 + \alpha^2 x^3 + \alpha^3 x^2 + \alpha^{44} x + \alpha^{34} \in F_{7^2}[x]$, choose $g(x) \in R_2[x]$ by $g(x) = \langle e_1 g_1(x), e_2 g_2(x)\rangle$ where $g_i(x) = \alpha^2 x^4 + \alpha^{18} x^3 + \alpha^3 x^2 + \alpha^3 3x + 4 \in F_{7^2}$. Clearly the code generated by C is $[10, 6, 4]$ from which $\phi(C)$ has the parameter $[20, 12, 4]$ and $C^o = [10, 4, 6]$ from which $\phi(C^o) = [20, 8, 6]$. Also, $Hull(C) = \{0\}$ being that $\dim(\phi(Hull(C))) = 0$. Hence we have a EAQECC as $[[20, 8, 6, 12]]$ over and $[[14, 12, 4, 8]]$ $\mathbb{F}_{7^2}$.*

Table 1. Comparison of new EAQECCs with existing EAQECCs over $\mathbb{F}_q$

Length n	$x^n - a(x)$	**Generator Polynomial(s)** $g_1(x)$	**Generator Polynomial(s)** $g_2(x)$	**Linear codes** $C : [n, k, d]$	$Hull(C)$	**New EAQECCs Codes**	**Existing EAQECCs**
10	$x^{10} - 1$	$(x+1)^5(x+4)$	$(x+4)^5(x+1)$	$[20, 8, 4]_5$	$[20, 2, 10]_5$	$[[20, 6, 4; 10]]_5$	$[[20, 5, 3; 5]]_5$[20]
6	$x^6 - 5x + 1$	$(x^2 + 2x + 4)$ $(x + 3)$	$(x^2 + 3x + 4)$	$[12, 5, 4]_5$	$[12, 1, 12]_5$	$[[12, 6, 4; 4]]_5$	$[[12, 3, 3; 3]]5$[20]
9	$x^9 - 4x^5 +$ $2x^3 + x + 1$	$(x + 3)^2$	$(x + 3)^2$ $(x^2 + 6x + 4)$	$[18, 12, 5]_7$	$[18, 1, 17]_7$	$[[18, 11, 5; 5]]_7$	$[[18, 9, 3; 3]]_7$[20]
15	$x^{15} - x^{12} +$ $2x^7 + x^2 + x$ $+4$	$(x + 2)^2$ $(x^3 + 3x^2 + 2x$ $+5)$	$(x + 1)$ $(x + 5)$ $(x^3 + 3x^2 + 2x$ $+5)$	$[30, 20, 6]_7$	$[30, 1, 28]_7$	$[[30, 19, 6; 9]]_7$	$[[30, 15, 3; 5]]_7$[20]

Table 1 compares the new EAQECCs constructed in this work with the best existing codes over $\mathbb{F}_q$. It shows that our constructions achieve superior parameters, confirming polycyclic codes as a strong framework for high-performance EAQECCs.

5. Conclusion

In this paper, the new entanglement-assisted quantum error-correcting codes are generated from the constructed polycyclic codes over the ring $R_l = \frac{\mathbb{F}_q[w]}{\langle w^l-1\rangle}$. Further, we established that the generated EAQECC are optimal.

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