

# On the spectrum of some cyclically and transitively oriented $\mathcal{H}$ -designs

Research Article

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**Abstract:** This paper investigates oriented  $\mathcal{H}$ -designs on complete uniform hypergraphs of rank 3, focusing in particular on the spectrum of existence and on the construction of some cyclically and transitively oriented  $P^{(3)}(2, 4)$ -designs and  $P^{(3)}(1, 5)$ -designs, namely  $BCP^{(3)}(2, 4)$ -designs and  $BCP^{(3)}(1, 5)$ -designs for cyclically oriented ones,  $BTP^{(3)}(2, 4)$ -designs and  $BTP^{(3)}(1, 5)$ -designs for transitively oriented ones. In the appendix, we provide the python code to obtain the explicit realization of the  $BCP^{(3)}(2, 4)$ -designs on  $v$  vertices. Moreover, the structure of this algorithm, with suitable modifications, can be generalized to the other three structures as well.

2020 MSC: 05C51, 05B05

Keywords:  $G$ -designs,  $\mathcal{H}$ -designs, Uniform hypergraphs

## 1. Introduction

The present work focuses on the study of cyclic and transitive designs, with particular attention devoted to the combinatorial structures denoted by  $P^{(3)}(2, 4)$ -designs and  $P^{(3)}(1, 5)$ -designs, defined over a set of vertex cardinality  $v$ . The main aim is to provide a comprehensive and rigorous analysis of the existence spectrum of such configurations, accompanied by the development of constructive methodologies, some of which are computationally efficient and amenable to algorithmic implementation. This article is based on results developed during the master's thesis work of the first author ([4]), whose goal was the study of the spectrum of block cyclic  $P^{(3)}(2, 4)$ -designs, a particular class of oriented  $\mathcal{H}$ -designs. More

\* The author is partially supported by Gnsaga of Indam and by Progetto Piaceri 2024-26, Università di Catania.

\*\* The author is partially supported by Gnsaga of Indam and by Progetto Piaceri 2024-26, Università di Catania. Guardo's research is also partially supported by PRIN 2022, "0-dimensional schemes, Tensor Theory and applications" - funded by the European Union Next Generation EU, Mission 4, Component 2 – CUP: E53D23005670006. Noemi Maria Dongola, Mario Gionfriddo, Elena Guardo (Corresponding Author), Giorgio Maria Rizzo; Dipartimento di Matematica e Informatica, University of Catania, Italy (email: noemi.dongola@phd.unict.it, mario.gionfriddo@unict.it, guardo.dmi.unict.it, giorgio.rizzo@phd.unict.it).

generally, this study falls within the context of the Woolhouse problem (1844) ([10]), a classical problem in design theory. Oriented  $\mathcal{H}$ -designs represent a specific case of this problem, in which an additional constraint is introduced: orientation. A triple can be oriented in essentially two ways: cyclically or transitively. If a cyclic orientation is considered for a  $P^{(3)}(2, 4)$ -design, one obtains a configuration known as a block cyclic  $P^{(3)}(2, 4)$ -design, whose existence spectrum is the subject of this study. In this article, we propose to extend the analysis to another family of oriented  $\mathcal{H}$ -designs, namely block cyclic  $P^{(3)}(1, 5)$ -designs. Furthermore, the same structures have also been studied under the assumption of transitive orientation, giving rise to block transitive  $P^{(3)}(2, 4)$ -designs and block transitive  $P^{(3)}(1, 5)$ -designs.

After introducing some definitions and notation, in chapter 2 we undertake the determination of the spectrum associated with  $P_3(v)$ -designs, a foundational step that is instrumental in addressing the base cases concerning cyclic  $P^{(3)}(2, 4)$ -designs. Chapter 3 is entirely devoted to an in-depth analysis of the latter concepts, approached from both a theoretical and a computational perspective. In addition to thoroughly determining the associated spectrum, several explicit theoretical realizations are presented, carefully selected based on their computational tractability. Among these, some have been implemented in python to facilitate practical exploration; the corresponding code, along with illustrative examples, is provided in the appendix to offer the reader a more hands-on understanding of the results. In chapter 4, we proceed analogously with the class of cyclic  $P^{(3)}(1, 5)$ -designs, for which we derive the relevant spectrum and propose viable methods of construction. Lastly, in chapter 5, we turn our attention to the transitive variants of the aforementioned structures, namely the  $P^{(3)}(2, 4)$ -designs and  $P^{(3)}(1, 5)$ -designs. For these, we investigate both the spectrum and generation strategies through a series of structural observations, thereby furnishing a unified and exhaustive account of all design families under consideration. First, we introduce the notion of orientation in hypergraphs. In particular, we need to determine how many possible ways a triple can be oriented. In general, two types of triples can be formed: cyclic and transitive ([3], [9]). We also provide explicit constructions, practical strategies for generating pure  $BCP^{(3)}(2, 4)$ -designs, and, at the end of the paper, we give a python algorithm for their implementation starting from any number of vertices (see 6).

**Definition 1.1.** *Let  $X$  be a set of  $v$  elements. A cyclic triple of  $X$  is defined as a set of three ordered pairs of distinct elements of  $X$  of the form  $(x, y), (y, z), (z, x)$ . Thus, we can define a generic cyclic triple as follows:*

$$[x, y, z] = [y, z, x] = [z, x, y] = \{(x, y); (y, z); (z, x)\}.$$

All three representations identify the same ordered pairs, that is, the same cycle; in fact, the three representations differ by cyclic permutations, that is, the elements are rotated cyclically while preserving the order in which  $x$  precedes  $y$ ,  $y$  precedes  $z$ , and  $z$  precedes  $x$ . Instead, the triple  $[x, z, y]$  is clearly distinct from the previous ones, as it identifies different pairs, namely  $(x, z), (z, y), (y, x)$ .

**Definition 1.2.** *Let  $X$  be a set of  $v$  elements. A transitive triple of  $X$  is defined as a set of three ordered pairs of distinct elements of  $X$  of the form  $(x, y), (y, z), (x, z)$ . Thus, we can define a generic transitive triple as follows:*

$$[x, y, z] = \{(x, y); (y, z); (x, z)\}.$$

The name originates from the transitive property, which in a certain sense is applied to the triple according to the order in which its elements appear. This makes it possible to identify the ordered pairs contained within the triples, that is, the pairs formed by selecting its elements two at a time while preserving their original order. For this reason, the order of the components is extremely important; therefore, transitive triples, unlike cyclic ones, have a unique representation. It can also be observed that for every cyclic or transitive triple, it is possible to define its inverse.

**Definition 1.3.** *Given a cyclic triple  $[x, y, z]$ , we define its inverse cyclic triple as the triple obtained by reversing any two consecutive elements, i.e.:*

$$[y, x, z] = [x, z, y] = [z, y, x] = \{(y, x); (x, z); (z, y)\}.$$

**Definition 1.4.** Given a transitive triple  $[x, y, z]$ , we define its inverse transitive triple as the triple obtained by reversing the first and third elements, i.e.:

$$[z, y, x] = \{(z, y); (y, x); (z, x)\}.$$

We now introduce the definitions of oriented  $\mathcal{G}$ -design and  $\mathcal{H}$ -design ([6]).

**Definition 1.5.** Given a digraph  $\mathcal{G}$ , an oriented  $\mathcal{G}$ -design of order  $v$  is a decomposition of the complete oriented graph  $\mathcal{K}_v$ , namely a partition of the set of ordered pairs of  $\mathcal{K}_v$  into classes, each of which generates a digraph isomorphic to  $\mathcal{G}$ .

Unlike oriented  $\mathcal{G}$ -designs, the definition of oriented  $\mathcal{H}$ -design depends on the orientation of the triples. Since a triple can be oriented either cyclically or transitively, an oriented complete hypergraph  $\mathcal{K}_v^{(3)}$  may consist exclusively of cyclic triples, exclusively of transitive triples, or a combination of both. For convenience, we will refer to a rank 3 hypergraph composed only of cyclic triples as a *cyclically oriented hypergraph*; similarly, a rank 3 hypergraph composed only of transitive triples will be called a *transitively oriented hypergraph*. The third case, involving “hybrid” hypergraphs, lies beyond the scope of this study. Therefore, we shall provide the following definition, which applies equivalently to both cyclic and transitive orientations.

**Definition 1.6.** Let  $\mathcal{H}^{(3)}$  be a cyclically (or transitively) oriented 3-uniform hypergraph. An oriented  $\mathcal{H}^{(3)}$ -design is a decomposition of the cyclically (or transitively) oriented complete 3-uniform hypergraph  $\mathcal{K}_v^{(3)}$ , namely a partition of the set of cyclic (or transitive) triples of  $\mathcal{K}_v^{(3)}$  into classes, each of which generates a 3-uniform hypergraph isomorphic to  $\mathcal{H}^{(3)}$ .

## 2. Spectrum of $P_3(v)$ -designs

The creation of the  $P_3(v)$ -design of order  $v$  is required to investigate the spectrum of  $BCP^{(3)}(2, 4)$ -designs and their related construction.

**Definition 2.1.** A  $P_3$  is a digraph  $(X, E)$  consisting of three vertices  $x, y, z$  and two directed edges  $(x, y), (z, y)$ . This digraph will be denoted by  $[x, (y), z]$ . The vertex  $y$  is referred to as the central vertex, while the other ones are called end points.

**Definition 2.2.** A  $P_3(v)$ -design of order  $v$  is a decomposition of the complete oriented graph  $\mathcal{K}_v$  into digraphs isomorphic to  $P_3$ .

**Theorem 2.3.** Let  $\Sigma = (X, \mathcal{B})$  be a  $P_3(v)$ -design of order  $v$ . Then:

1.  $|\mathcal{B}| = \frac{v(v-1)}{2}$ ;
2.  $v \equiv 1, \text{ mod } 2, v \geq 3$ .

**Proof.** (i) All ordered pairs on  $v$  elements must be grouped in pairs; therefore, we have:

$$|\mathcal{B}| = \frac{2\binom{v}{2}}{2} = \binom{v}{2} = \frac{v(v-1)}{2}.$$

(ii) Since  $|\mathcal{B}| = \frac{v(v-1)}{2} \in \mathbb{N}$ , it follows that  $v \equiv 0$  or  $v \equiv 1, \text{ mod } 2$ . However, for each vertex  $x \in X$ , the ordered pairs of the form  $(\cdot, x)$  must be assigned such that  $x$  appears as the central vertex in some  $P_3$ ; hence, its in-degree must be even. As there are  $v - 1$  remaining vertices to assign as end points, it must be that for all  $x \in X, d_i(x) = \frac{v-1}{2} \in \mathbb{N}$ . Thus,  $v - 1$  is even, i.e.  $v$  is odd. We therefore conclude that  $v \equiv 1, \text{ mod } 2$  and  $v \geq 3$ .  $\square$

**Theorem 2.4.** A  $P_3(v)$ -design of order  $v$  exists for every  $v \equiv 1, \pmod 2, v \geq 3$ .

**Proof.** Let  $X = \mathbb{Z}_{2h+1} = \{0, 1, 2, \dots, 2h\}$  and define  $X_i = \mathbb{Z}_{2h+1} \setminus \{i\}$ . For each  $i = 0, 1, 2, \dots, 2h$ , consider a 1-factor of  $X_i$  constructed as follows:

$$F(X_i) = \{(0, 1); (2, 3); \dots; (i - 1, i + 1); \dots; (2h - 1, 2h)\}.$$

We then define the following set of blocks:

$$\mathcal{B} = \{[\alpha, i, \beta] \mid (\alpha, \beta) \in F(X_i), i \in \mathbb{Z}_{2h+1}\}.$$

Then  $\Sigma = (X, \mathcal{B})$  is a  $P_3(v)$ -design of order  $v = 2h + 1$ .

*Existence:* Any ordered pair  $(x, y)$  must appear either as  $(\alpha, i)$  or  $(i, \beta)$ . In both cases, by construction, there exists at least one block in  $\mathcal{B}$  containing it.

*Uniqueness:* It is immediate to verify that:

$$|\mathcal{B}| = |X||F(X_i)| = (2h + 1)h = \frac{v(v - 1)}{2}.$$

□

We shall now first examine the cyclically oriented  $\mathcal{H}$ -designs, and subsequently the transitively oriented ones, as the latter can be derived from the former.

### 3. Spectrum of $BCP^{(3)}(2, 4)$ -designs

We are now equipped with all the fundamental elements and concepts necessary to study the spectrum of  $BCP^{(3)}(2, 4)$ -designs, which are particular oriented  $\mathcal{H}^{(3)}$ -designs (see [6], [8]).

**Definition 3.1.** A block cyclic  $P^{(3)}(2, 4)$  is a cyclically oriented hypergraph  $(X, \mathcal{B})$  consisting of four vertices  $x, y, z, t$  and two cyclic triples  $[x, y, z], [x, y, t]$ . This system will be denoted as  $BCP^{(3)}(2, 4) = [z, (x, y), t]$  and identifies the configuration in which the central oriented edge is determined by the ordered pair  $(x, y)$ .

In a  $BCP^{(3)}(2, 4)$ , the two cyclic triples share an ordered pair, which constitutes the main challenge in the constructions that we will examine later. Indeed, all such constructions are based on the necessity to pair each available cyclic triple appropriately.

**Definition 3.2.** A  $BCP^{(3)}(2, 4)$ -design of order  $v$  is a decomposition of the complete cyclic hypergraph  $\mathcal{K}_v^{(3)}$  into hypergraphs isomorphic to  $BCP^{(3)}(2, 4)$ .

Thus, for every cyclic triple on  $v$  elements, there exists one and only one block in which that triple appears. Note that, given a  $BCP^{(3)}(2, 4)$  block defined by two cyclic triples, it is possible to define its *opposite block*, obtained by taking the inverse cyclic triples of the original ones. According to Definition 1.4, the opposite block can be constructed by reversing the two vertices that form the central oriented edge.

**Definition 3.3.** Given a  $BCP^{(3)}(2, 4)$ -design  $\Sigma = (X, \mathcal{B})$  on  $v$  vertices, the *purity index*, denoted by  $\gamma$ , is defined as the minimum number of  $BCP^{(3)}(2, 4)$  opposite blocks that appear in  $\Sigma$ .

**Remark 3.4.** Note that  $\gamma$  is always an even number.

**Definition 3.5.** A  $BCP^{(3)}(2, 4)$ -design is said to be *pure* if  $\gamma = 0$  and *non-pure* if  $\gamma \neq 0$ .

In what follows, our attention will be specifically devoted to the study of pure  $BCP^{(3)}(2, 4)$ -designs, that is, to the particular case in which the parameter  $\gamma$  is equal to zero. This setting is of primary interest in our investigation, as it gives rise to richer and structurally more intricate constructions from a combinatorial point of view, offering a broader scope for analysis and theoretical development. By contrast, when  $\gamma \neq 0$ , the resulting constructions become significantly simpler and more straightforward to derive, thus rendering them less relevant and less stimulating for the objectives of our study, which aim to explore the most complex and meaningful instances of these combinatorial designs.

**Theorem 3.6.** *Given a  $BCP^{(3)}(2, 4)$ -design of order  $v$ , the following hold:*

1.  $|\mathcal{B}| = \frac{v(v-1)(v-2)}{6}$ ;
2.  $v \geq 4$ .

**Proof.** To enumerate all cyclic triples on  $v$  elements, observe that each unordered triple generates two cyclic triples, i.e. the cyclically ordered one and its inverse. Hence, the number of blocks in a complete cyclically oriented hypergraph  $\mathcal{K}_v^{(3)}$  is  $2\binom{v}{3}$ . Since each  $BCP^{(3)}(2, 4)$  contains exactly two cyclic triples, we obtain:

$$|\mathcal{B}| = \frac{2\binom{v}{3}}{2} = \binom{v}{3} = \frac{v(v-1)(v-2)}{6} \in \mathbb{N},$$

from which it follows that  $v \geq 4$ . □

**Theorem 3.7.** *A  $BCP^{(3)}(2, 4)$ -design of order  $v$  exists for every  $v \geq 4$ .*

**Proof.** Starting from four base cases, we provide two constructions: one covering the spectrum of even values of  $v$ , and the other for odd values of  $v$ .

**Case  $v = 4$**

Let  $X = \{0, 1, 2, 3\}$  and  $\mathcal{B} = \{[2, (0, 1), 3], [3, (0, 2), 1], [2, (3, 1), 0], [0, (3, 2), 1]\}$ . One can easily verify that  $\Sigma = (X, \mathcal{B})$  is a  $BCP^{(3)}(2, 4)$ -design of order 4.

**Case  $v = 5$**

Let  $X = \mathbb{Z}_5$  and define the block set as  $\mathcal{B} = \{[i-1, (i, i+2), i+1], [i-1, (i, i+3), i+1] \mid i \in \mathbb{Z}_5\}$ . It is straightforward to verify that  $\Sigma = (X, \mathcal{B})$  is a  $BCP^{(3)}(2, 4)$ -design of order 5.

**Case  $v = 6$**

Let  $X = \mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$  and define  $X' = X \cup \{5\}$ , where  $5 \notin X$ . Let  $\Sigma_1 = (X, \mathcal{B}_1)$  be a  $BCP^{(3)}(2, 4)$ -design of order 5 and let  $\Sigma_2 = (X, \mathcal{B}_2)$  be a  $P_3(5)$ -design. Construct the following blocks:

$$\Pi = \{[x, (y, 5), z] \mid [x, (y), z] \in \mathcal{B}_2\}.$$

It is easy to verify that  $\Sigma = (X', \mathcal{B}')$  is a  $BCP^{(3)}(2, 4)$ -design of order 6, where  $\mathcal{B}' = \mathcal{B}_1 \cup \Pi$ .

**Case  $v = 7$**

Let  $X = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$  and define  $X' = X \cup \{6\}$ , where  $6 \notin X$ . Let  $\Sigma = (X, \mathcal{B})$  be a  $BCP^{(3)}(2, 4)$ -design of order 6. Consider the factor  $F(X) = \{(0, 1), (2, 3), (4, 5)\}$  of  $X$  and let  $\mathcal{D} = \{1, 3, 5\}$  be the set of odd numbers less than 6. Construct the following blocks:

$$\begin{aligned} \Pi_1 &= \{[x, (y + i, 6), y] \mid (x, y) \in F(X), i = 1, 2, 3\}, \\ \Pi_2 &= \{[i - 1, (6, i), i + 1]; [i - 1, (6, i + 1), i] \mid i \in \mathcal{D}\}. \end{aligned}$$

Every element of the previous blocks, except 6, is computed in  $\mathbb{Z}_6$ . It can be verified that  $\Sigma' = (X', \mathcal{B}')$  is a  $BCP^{(3)}(2, 4)$ -design of order 7, where  $\mathcal{B}' = \mathcal{B} \cup \Pi_1 \cup \Pi_2$ .

**Construction for  $v = 2h \longrightarrow v + 4 = 2h + 4 (h \geq 2)$  :**

Let  $X_1 = \mathbb{Z}_{2h} = \{0, 1, \dots, 2h - 1\}$ ,  $X_2 = \{2h, 2h + 1, 2h + 2, 2h + 3\}$ . We consider:

$$\begin{aligned} \Sigma_1 &= (X_1, \mathcal{B}_1) = BCP^{(3)}(2, 4)\text{-design of order } 2h; \\ \Sigma_2 &= (X_2, \mathcal{B}_2) = BCP^{(3)}(2, 4)\text{-design of order } 4. \end{aligned}$$

For each set  $X_1$  and  $X_2$  we consider the two factorizations of the form:

$$\begin{aligned} F(X_1) &= \{(0, 1); (2, 3); \dots; (2h - 4, 2h - 3); (2h - 2, 2h - 1)\}; \\ F'(X_1) &= \{(0, h); (1, h + 1); \dots; (h - 1, 2h - 1)\}; \\ F(X_2) &= \{(2h, 2h + 1); (2h + 2, 2h + 3)\}; \\ F'(X_2) &= \{(2h, 2h + 2); (2h + 1, 2h + 3)\}. \end{aligned}$$

For each pair  $(x', y') \in \mathcal{P}_2(X_2)$ , we construct the following blocks:

$$\Pi_1 = \{[\alpha, (x', y'), \beta]; [\alpha', (y', x'), \beta'] \mid (\alpha, \beta) \in F(X_1), (\alpha', \beta') \in F'(X_1)\}.$$

On the contrary, for each pair  $(x, y) \in \mathcal{P}_2(X_1)$ , we construct the following blocks:

$$\Pi_2 = \{[\gamma, (x, y), \delta]; [\gamma', (y, x), \delta'] \mid (\gamma, \delta) \in F(X_2), (\gamma', \delta') \in F'(X_2)\}.$$

Then  $\Sigma = (X, \mathcal{B})$  is a  $BCP^{(3)}(2, 4)$ -design of order  $2h + 4$ , where:

- $X = X_1 \cup X_2$ ;
- $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \bigcup_{(x', y') \in \mathcal{P}_2(X_2)} \Pi_1 \cup \bigcup_{(x, y) \in \mathcal{P}_2(X_1)} \Pi_2$ .

*Existence.* Let  $(x, y, z)$  be a cyclic triple of  $\Sigma$ ; the following cases may occur:

1. if  $x, y, z \in X_1$ , then the triple is contained in exactly one block of  $\Sigma_1$ ;
2. if  $x, y, z \in X_2$ , then the triple is contained in exactly one block of  $\Sigma_2$ ;
3. if  $x, y \in X_1$  and  $z \in X_2$ , then the triple is contained in at least one block of  $\Pi_2$  since  $(x, y) \in \mathcal{P}_2(X_1)$ ;
4. if  $x \in X_1$  and  $y, z \in X_2$ , then the triple is contained in at least one block of  $\Pi_1$  since  $(y, z) \in \mathcal{P}_2(X_2)$ ;
5. all other cases follow from *iii.* and *iv.*

*Uniqueness.* First note that for  $v = 2h + 4$ , the number of blocks is given by:

$$\begin{aligned} |\mathcal{B}| &= \frac{v(v - 1)(v - 2)}{6} = \frac{(2h + 4)(2h + 3)(2h + 2)}{6} = \frac{(2h + 4)(2h + 3)(h + 1)}{3} = \\ &= \frac{(4h^2 + 14h + 12)(h + 1)}{3} = \frac{4h^3 + 18h^2 + 26h + 12}{3}. \end{aligned}$$

Verifying, we obtain:

$$\begin{aligned}
 |\mathcal{B}| &= |\mathcal{B}_1| + |\mathcal{B}_2| + \left| \bigcup_{(x',y') \in \mathcal{P}_2(X_2)} \Pi_1 \right| + \left| \bigcup_{(x,y) \in \mathcal{P}_2(X_1)} \Pi_2 \right| = \\
 &= \frac{2h(2h-1)(2h-2)}{6} + 4 + 2h \binom{4}{2} + 4 \binom{2h}{2} = \\
 &= \frac{h(2h-1)(2h-2)}{3} + 4 + 12h + 4h(2h-1) = \\
 &= \frac{h(4h^2 - 6h + 2)}{3} + 8h^2 + 8h + 4 = \\
 &= \frac{4h^3 + 18h^2 + 26h + 12}{3}.
 \end{aligned}$$

**Construction for  $v = 2h + 1 \longrightarrow v + 4 = 2h + 5$  ( $h \geq 2$ ) :**

Let  $X_1 = \mathbb{Z}_{2h+1} = \{0, 1, \dots, 2h\}$ ,  $X_2 = \{2h + 1, 2h + 2, 2h + 3, 2h + 4\}$ . We consider:

$$\begin{aligned}
 \Sigma_1 &= (X_1, \mathcal{B}_1) = BCP^{(3)}(2, 4)\text{-design of order } 2h + 1; \\
 \Sigma_2 &= (X_2, \mathcal{B}_2) = BCP^{(3)}(2, 4)\text{-design of order } 4.
 \end{aligned}$$

Given  $\Sigma_3 = (X_1, \mathcal{B}) = P_3(2h + 1)$ -design, for every  $\alpha \in X_2$ , consider the following blocks:

$$\Pi_\alpha = \{[a, (b, \alpha), c] \mid [a, (b), c] \in \mathcal{B}\}.$$

Consider the factorizations of  $X_1 \setminus \{0\}$  as follows:

$$\begin{aligned}
 F(X_1 \setminus 0) &= \{(1, 2); (3, 4); \dots; (2h - 3, 2h - 2); (2h - 1, 2h)\}; \\
 F'(X_1 \setminus 0) &= \{(1, h + 1); (2, h + 2); \dots; (h, 2h)\}.
 \end{aligned}$$

Then, for each pair  $(x, y) \in \mathcal{P}_2(X_2)$ , we define:

$$\Pi = \{[\alpha, (x, y), \beta]; [\alpha', (y, x), \beta'] \mid (\alpha, \beta) \in F(X_1 \setminus \{0\}), (\alpha', \beta') \in F'(X_1 \setminus \{0\})\}.$$

The cyclic triples containing element 0 remain unaccounted, thus we define the final set of blocks:

$$\begin{aligned}
 \Delta &= \{[2h + 2, (0, 2h + 1), 2h + 3]; [2h + 1, (2h + 4, 0), 2h + 3]; \\
 &[2h + 3, (0, 2h + 2), 2h + 4]; [2h + 2, (2h + 1, 0), 2h + 4]; \\
 &[2h + 1, (0, 2h + 3), 2h + 2]; [2h + 2, (0, 2h + 4), 2h + 3]\}.
 \end{aligned}$$

Then  $\Sigma = (X, \mathcal{B})$  is a  $BCP^{(3)}(2, 4)$ -design of order  $2h + 5$ , where:

- $X = X_1 \cup X_2$ ;
- $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \bigcup_{\alpha \in X_2} \Pi_\alpha \cup \Pi \cup \Delta$ .

*Existence.* Let  $(x, y, z)$  be a cyclic triple of  $\Sigma$ ; we distinguish the following cases:

1. if  $x, y, z \in X_1$ , then the triple is contained in exactly one block of  $\Sigma_1$ ;
2. if  $x, y, z \in X_2$ , then the triple is contained in exactly one block of  $\Sigma_2$ ;
3. if  $x, y \in X_1$  and  $z \in X_2$ , then the cycle includes only one element from  $X_2$ , thus the triple is contained in at least one block of  $\Pi_z$ ;

4. if  $x \in X_1$  and  $y, z \in X_2$ , then the cycle includes a pair from  $X_2$ , thus the triple is contained in at least one block of  $\Pi$ ;
5. if  $x = 0$  and  $y, z \in X_2$ , then the triple is necessarily contained in at least one block of  $\Delta$ ;
6. all other cases fall under the ones above.

*Uniqueness.* First note that for  $v = 2h + 5$ , the number of blocks is given by:

$$\begin{aligned}
 |\mathcal{B}| &= \frac{v(v-1)(v-2)}{6} = \frac{(2h+5)(2h+4)(2h+3)}{6} = \frac{(2h+5)(h+2)(2h+3)}{3} = \\
 &= \frac{(4h^2 + 16h + 15)(h+2)}{3} = \frac{4h^3 + 24h^2 + 47h + 30}{3}.
 \end{aligned}$$

Verifying, we obtain:

$$\begin{aligned}
 |\mathcal{B}| &= |\mathcal{B}_1| + |\mathcal{B}_2| + \left| \bigcup_{\alpha \in X_2} \Pi_\alpha \right| + |\Pi| + |\Delta| = \\
 &= \frac{(2h+1)2h(2h-1)}{6} + 4 + 4 \cdot \frac{(2h+1)2h}{2} + \binom{4}{2} \cdot 2h + 6 = \\
 &= \frac{h(4h^2-1)}{3} + 10 + 8h^2 + 4h + 12h = \frac{4h^3 - h + 24h^2 + 48h + 30}{3} = \\
 &= \frac{4h^3 + 24h^2 + 47h + 30}{3}.
 \end{aligned}$$

□

### 3.1. Other constructions

The constructions used for the base cases  $v = 6$  and  $v = 7$  can be appropriately generalized. Specifically, the construction corresponding to the case  $v = 6$  naturally applies whenever one aims to construct a  $BCP^{(3)}(2, 4)$ -design on an even value of  $v$ , starting from the immediately preceding odd value  $v - 1$ . In contrast, the construction developed for the case  $v = 7$  remains valid whenever the goal is to build a  $BCP^{(3)}(2, 4)$ -design on an odd value  $v$ , using the corresponding even value  $v - 1$  as a basis.

These two recursive strategies prove particularly advantageous in the context where a  $BCP^{(3)}(2, 4)$ -design has already been established for a given integer  $v$ , and the aim is to extend this knowledge to construct the design on  $v + 1$ . In what follows, we present these constructions in detail, emphasizing not only their practical applicability in facilitating such an extension but also their structural effectiveness, which enables a systematic and coherent approach to the incremental development of these combinatorial objects.

#### Construction for $v = 2h + 1 \longrightarrow v + 1 = 2h + 2$ ( $h \geq 2$ ):

Let  $X = \mathbb{Z}_{2h+1} = \{0, 1, \dots, 2h\}$ . We consider:

$$\begin{aligned}
 \Sigma_1 &= (X, \mathcal{B}) = BCP^{(3)}(2, 4)\text{-design of order } 2h + 1; \\
 \Sigma_2 &= (X, \mathcal{T}) = P_3(2h + 1)\text{-design.}
 \end{aligned}$$

The additional blocks are defined by the following set:

$$\Pi = \{[x, (y, \infty), z] \mid [x, (y), z] \in \mathcal{T}\}.$$

Then  $\Sigma' = (X', \mathcal{B}')$  is a  $BCP^{(3)}(2, 4)$ -design on  $v = 2h + 2$ , where:

- $X' = X \cup \{\infty\}$ , with  $\infty \notin X$ ;
- $\mathcal{B}' = \mathcal{B} \cup \Pi$ .

*Existence.* Let  $(x, y, z)$  be a cyclic triple in  $\Sigma'$  and consider the following cases:

1. if  $x, y, z \in X$ , then the cyclic triple will be contained in exactly one block of  $\mathcal{B}$ ;
2. if  $x, y \in X$  and  $z = \infty$ , then the cyclic triple will be contained in at least one block of  $\Pi$ .

*Uniqueness.* As usual, observe that for  $v = 2h + 2$ , the number of blocks is:

$$\begin{aligned} |\mathcal{B}'| &= \frac{v(v-1)(v-2)}{6} = \frac{(2h+2)(2h+1)2h}{6} = \\ &= \frac{(4h^2+6h+2)h}{3} = \frac{4h^3+6h^2+2h}{3}. \end{aligned}$$

Verifying, we obtain:

$$\begin{aligned} |\mathcal{B}'| &= |\mathcal{B}| + |\Pi| = \frac{(2h+1)2h(2h-1)}{6} + \frac{(2h+1)2h}{2} = \\ &= \frac{h(4h^2-1)}{3} + h(2h+1) = \frac{4h^3-h+6h^2+3h}{3} = \frac{4h^3+6h^2+2h}{3}. \end{aligned}$$

**Construction for  $v = 2h \longrightarrow v + 1 = 2h + 1$  ( $h \geq 2$ ):**

Let  $X = \mathbb{Z}_{2h} = \{0, 1, \dots, 2h - 1\}$ . We consider:

$$\begin{aligned} \Sigma &= (X, \mathcal{B}) = BCP^{(3)}(2, 4)\text{-design of order } 2h; \\ \mathcal{D} &= \{1, 3, 5, \dots, 2h - 1\} \subseteq \mathbb{Z}_{2h}, \text{ i.e. the odd integers less than } 2h + 1. \end{aligned}$$

Now, we add the following 1-factor of  $X$ :

$$F(X) = \{(0, 1); (2, 3); \dots; (2h - 2, 2h - 1)\}.$$

The additional blocks are given by the sets:

$$\begin{aligned} \Pi_1 &= \{[x, (y + i, \infty), y] \mid (x, y) \in F(X), i = 0, \dots, 2h - 4\}; \\ \Pi_2 &= \{[i - 1, (\infty, i), i + 1]; [i - 1, (\infty, i + 1), i] \mid i \in \mathcal{D}\}. \end{aligned}$$

It follows that  $\Sigma' = (X', \mathcal{B}') = BCP^{(3)}(2, 4)$ -design on  $v = 2h + 1$ , where:

- $X' = X \cup \{\infty\}$ , with  $\infty \notin X$ ;
- $\mathcal{B}' = \mathcal{B} \cup \Pi_1 \cup \Pi_2$ .

*Existence.* Let  $(x, y, z)$  be a cyclic triple in  $\Sigma'$ , and consider the following cases:

1. if  $x, y, z \in X$ , then the cyclic triple will be contained in exactly one block of  $\mathcal{B}$ ;
2. if  $x, y \in X$  and  $z = \infty$ , then the cyclic triple will be contained in at least one block of  $\Pi_1$  or  $\Pi_2$ .

*Uniqueness.* As usual, observe that for  $v = 2h + 1$ , the number of blocks is:

$$\begin{aligned} |\mathcal{B}'| &= \frac{v(v-1)(v-2)}{6} = \frac{(2h+1)2h(2h-1)}{6} = \\ &= \frac{(4h^2-1)h}{3} = \frac{4h^3-h}{3}. \end{aligned}$$

Verifying, we obtain:

$$\begin{aligned} |\mathcal{B}'| &= |\mathcal{B}| + |\Pi_1| + |\Pi_2| = \frac{2h(2h-1)(2h-2)}{6} + h(2h-3) + 2h = \\ &= \frac{h(4h^2-6h+2)}{3} + 2h^2 - 3h + 2h = \frac{4h^3-6h^2+2h}{3} + 2h^2 - h = \\ &= \frac{4h^3-6h^2+2h+6h^2-3h}{3} = \frac{4h^3-h}{3}. \end{aligned}$$

A final remark concerns the construction for even  $v$ , where an additional and particularly useful simplification becomes available: in these cases, the construction process can also be successfully carried out without the need to explicitly construct the  $P_3(v)$ -design on  $v - 1$ , which is otherwise required in the recursive approach. This alternative method allows for bypassing the intermediate step, significantly streamlining the overall procedure while still ensuring that the resulting design satisfies all the required structural properties and combinatorial constraints.

**Construction for  $v = 2h + 1 \longrightarrow v + 1 = 2h + 2$  ( $h \geq 2$ ):**

Let  $X = \mathbb{Z}_{2h+1} = \{0, 1, \dots, 2h\}$ . We consider:

$$\Sigma = (X, \mathcal{B}) = BCP^{(3)}(2, 4)\text{-design of order } 2h + 1.$$

The additional blocks are given by the following set:

$$\Pi = \{[i + 1, (\infty, i), i + 2]; [i + 3, (\infty, i), i + 4]; \dots; [2h - 1, (\infty, i), 2h] \mid i \in \mathbb{Z}_{2h+1}\}.$$

It follows that  $\Sigma' = (X', \mathcal{B}') = BCP^{(3)}(2, 4)$ -design on  $v = 2h + 2$ , where:

- $X' = X \cup \{\infty\}$ , with  $\infty \notin X$ ;
- $\mathcal{B}' = \mathcal{B} \cup \Pi$ .

*Existence.* Let  $(x, y, z)$  be a cyclic triple in  $\Sigma'$ , and consider the following cases:

1. if  $x, y, z \in X$ , then the cyclic triple will be contained in exactly one block of  $\Sigma$ ;
2. if  $x, y \in X$  and  $z = \infty$ , then the cyclic triple will be contained in at least one block of  $\Pi$ .

*Uniqueness.* For  $v = 2h + 2$ , the number of blocks is:

$$\begin{aligned} |\mathcal{B}'| &= \frac{v(v-1)(v-2)}{6} = \frac{(2h+2)(2h+1)2h}{6} = \\ &= \frac{(4h^2+6h+2)h}{3} = \frac{4h^3+6h^2+2h}{3}. \end{aligned}$$

Verifying, we obtain:

$$\begin{aligned} |\mathcal{B}'| &= |\mathcal{B}| + |\Pi| = \frac{(2h+1)2h(2h-1)}{6} + h(2h+1) = \\ &= \frac{h(4h^2-1)}{3} + 2h^2 + h = \frac{4h^3-h+6h^2+3h}{3} = \\ &= \frac{4h^3+6h^2+2h}{3}. \end{aligned}$$

This concludes the analysis of the spectrum of  $BCP^{(3)}(2, 4)$ -designs of order  $v$ . The study of their existence and construction is now complete.

#### 4. Spectrum of $BCP^{(3)}(1, 5)$ -designs

Now we aim to study another significant category of cyclically oriented  $\mathcal{H}^{(3)}$ -designs, i.e. the block cyclic  $P^{(3)}(1, 5)$ -designs.

**Definition 4.1.** A block cyclic  $P^{(3)}(1, 5)$  is a cyclically oriented hypergraph  $(X, \mathcal{B})$  consisting of five vertices  $x, y, z, t, w$  and two cyclic triples  $[x, y, z], [x, t, w]$  that share the vertex  $x$  in common. This system is denoted by  $BCP^{(3)}(1, 5) = [y, z, (x), t, w]$ , where the vertex  $x$  is the common (central) element in both triples.

Unlike the  $BCP^{(3)}(2, 4)$  systems, in the  $BCP^{(3)}(1, 5)$  ones the orientation of the two cyclic triples is not relevant, since changing the orientation does not affect the in-degree and out-degree of any vertex. This is due to the fact that the two cyclic triples share only a single vertex, rather than common edges (a more restrictive condition).

**Definition 4.2.** A  $BCP^{(3)}(1, 5)$ -design of order  $v$  is a decomposition of the complete cyclic hypergraph  $\mathcal{K}_v^{(3)}$  into hypergraphs isomorphic to  $BCP^{(3)}(1, 5)$ .

Thus, for every cyclic triple on  $v$  elements, there exists one and only one  $BCP^{(3)}(1, 5)$  block in which that triple appears. In this case, also, a  $BCP^{(3)}(1, 5)$ -design may contain the so-called opposite blocks. As in the previous setting, our goal is to investigate the spectrum of pure  $BCP^{(3)}(1, 5)$ -designs, namely those with  $\gamma = 0$  (i.e. free of opposite blocks).

**Theorem 4.3.** Given a  $BCP^{(3)}(1, 5)$ -design of order  $v$ , the following hold:

1.  $|\mathcal{B}| = \frac{v(v-1)(v-2)}{6}$ ;
2.  $v \geq 5$ .

**Proof.** The proof follows the same argument of the theorem 3.6. In fact, to enumerate all cyclic triples on  $v$  elements, observe that each unordered triple generates two cyclic triples, i.e. the cyclically ordered one and its inverse. Hence, the number of blocks in a complete cyclically oriented hypergraph  $\mathcal{K}_v^{(3)}$  is  $2\binom{v}{3}$ . Since each  $BCP^{(3)}(1, 5)$  contains exactly two cyclic triples, we obtain:

$$|\mathcal{B}| = \frac{2\binom{v}{3}}{2} = \binom{v}{3} = \frac{v(v-1)(v-2)}{6} \in \mathbb{N},$$

from which it follows that  $v \geq 5$ . □

**Theorem 4.4.** A  $BCP^{(3)}(1, 5)$ -design of order  $v$  exists for every  $v \geq 5$ .

**Proof.** Starting from a base case, we perform a construction which allows us to obtain a  $BCP^{(3)}(1, 5)$ -design on  $v + 1$  vertices from the previous one on  $v$  vertices.

**Case  $v = 5$**

Let  $X = \mathbb{Z}_5$  and define the block set as:

$$\mathcal{B} = \{[i + 4, i + 3, (i), i + 2, i + 1], [i + 4, i + 2, (i), i + 3, i + 1] \mid i \in \mathbb{Z}_5\}.$$

It is straightforward to verify that  $\Sigma = (X, \mathcal{B})$  is a  $BCP^{(3)}(1, 5)$ -design of order 5.

**Construction for  $v \rightarrow v + 1$  :**

**Case (1A):**  $v = 4h, h \geq 2$ .

Let be  $X = \mathbb{Z}_{4h} = \{0, 1, \dots, 4h - 1\}$  and  $X' = X \cup \{\infty\}$  such that  $\infty \notin X$ . We consider:

$$\Sigma = (X, \mathcal{B}) = BCP^{(3)}(1, 5)\text{-design of order } 4h.$$

We also consider a factorization  $\mathcal{F}$  of the set  $X$ . Since the cardinality of  $X$  is even, we have  $4h - 1$  factors such that  $|F_i| = 2h$ , for all  $i = 1, \dots, 4h - 1$ :

$$\mathcal{F} = \{F_1, F_2, \dots, F_{4h-1}\}.$$

If the generic factor is  $F_i = \{(a_{i,1}, a_{i,2}); (a_{i,3}, a_{i,4}); \dots; (a_{i,4h-1}, a_{i,4h})\}$ , then for all  $i = 1, \dots, 4h - 1$  (i.e. for each factor) we construct the following blocks like that:

$$\Gamma_i = \{[a_{i,1}, a_{i,2}, (\infty), a_{i,3}, a_{i,4}]; \dots; [a_{i,4h-3}, a_{i,4h-2}, (\infty), a_{i,4h-1}, a_{i,4h}]; [a_{i,4}, a_{i,3}, (\infty), a_{i,6}, a_{i,5}]; \dots; [a_{i,4h}, a_{i,4h-1}, (\infty), a_{i,2}, a_{i,1}]\}.$$

Let  $\Gamma = \bigcup_{i=1}^{4h-1} \Gamma_i$ , then  $\Sigma' = (X', \mathcal{B}')$  is a  $BCP^{(3)}(1, 5)$ -design of order  $4h + 1$ , where  $X' = X \cup \{\infty\}$  and  $\mathcal{B}' = \mathcal{B} \cup \Gamma$ .

*Existence.* Let  $(x, y, z)$  be a cyclic triple of  $\Sigma'$ , the following cases may occur:

1. if  $x, y, z \in X$ , then the triple is contained in exactly one block of  $\Sigma$ ;
2. if  $x, y \in X$  and  $z = \infty$ , then the triple is contained in at least one block of  $\Gamma$  since  $(x, y)$  will be in exactly one  $F_i \in \mathcal{F}$  of  $X$ ;
3. the remaining possible cases follow from the previous ones.

*Uniqueness.* First note that for  $v = 4h + 1$ , the number of blocks is given by:

$$|\mathcal{B}'| = \frac{v(v-1)(v-2)}{6} = \frac{(4h+1)4h(4h-1)}{6}.$$

Verifying, we obtain:

$$\begin{aligned} |\mathcal{B}'| &= |\mathcal{B}| + |\Gamma| = \frac{4h(4h-1)(4h-2)}{6} + \sum_{i=1}^{4h-1} |\Gamma_i| = \\ &= \frac{4h(4h-1)(4h-2)}{6} + (4h-1)2h = \\ &= \frac{(4h-1)(16h^2 + 4h)}{6} = \frac{(4h+1)4h(4h-1)}{6}. \end{aligned}$$

**Case (1B):**  $v = 4h + 1, h \geq 2$ .

Let be  $X = \mathbb{Z}_{4h+1} = \{0, 1, \dots, 4h\}$  and  $X' = X \cup \{\infty\}$  such that  $\infty \notin X$ . We consider:

$$\Sigma = (X, \mathcal{B}) = BCP^{(3)}(1, 5)\text{-design of order } 4h + 1.$$

Since the cardinality of  $X$  is odd, we consider a pseudo-factorization  $\mathcal{F}$  of the set  $X$ , in which we will have  $4h + 1$  factors such that  $|F_i| = 2h$ , for all  $i = 1, \dots, 4h + 1$ :

$$\mathcal{F} = \{F_1, F_2, \dots, F_{4h+1}\}.$$

Then we construct the same set  $\Gamma_i$ , for all  $i = 1, \dots, 4h + 1$  as the previous case:

$$\Gamma_i = \{[a_{i,1}, a_{i,2}, (\infty), a_{i,3}, a_{i,4}]; \dots; [a_{i,4h-3}, a_{i,4h-2}, (\infty), a_{i,4h-1}, a_{i,4h}]; [a_{i,4}, a_{i,3}, (\infty), a_{i,6}, a_{i,5}]; \dots; [a_{i,4h}, a_{i,4h-1}, (\infty), a_{i,2}, a_{i,1}]\}.$$

Let  $\Gamma = \bigcup_{i=1}^{4h+1} \Gamma_i$ , we obtain  $\Sigma' = (X', \mathcal{B}')$  is a  $BCP^{(3)}(1, 5)$ -design of order  $4h + 2$ , where  $X' = X \cup \{\infty\}$  and  $\mathcal{B}' = \mathcal{B} \cup \Gamma$ .

*Existence.* Let  $(x, y, z)$  be a cyclic triple of  $\Sigma'$ ; the following cases may occur:

1. if  $x, y, z \in X$ , then the triple is contained in exactly one block of  $\Sigma$ ;
2. if  $x, y \in X$  and  $z = \infty$ , then the triple is contained in at least one block of  $\Gamma$  since  $(x, y)$  will be in exactly one  $F_i \in \mathcal{F}$  of  $X$ ;
3. the remaining possible cases follow from the previous ones.

*Uniqueness.* First note that for  $v = 4h + 2$ , the number of blocks is given by:

$$|\mathcal{B}'| = \frac{v(v-1)(v-2)}{6} = \frac{(4h+2)(4h+1)4h}{6}.$$

Verifying, we obtain:

$$\begin{aligned} |\mathcal{B}'| &= |\mathcal{B}| + |\Gamma| = \frac{(4h+1)4h(4h-1)}{6} + \sum_{i=1}^{4h+1} |\Gamma_i| = \\ &= \frac{(4h+1)4h(4h-1)}{6} + (4h+1)2h = \\ &= \frac{(4h+1)(16h^2+8h)}{6} = \frac{(4h+2)(4h+1)4h}{6}. \end{aligned}$$

**Case (2A)**  $v = 2h$ ,  $h \geq 3$ ,  $h$  odd.

Let be  $X = \mathbb{Z}_{2h} = \{0, 1, \dots, 2h - 1\}$  and  $X' = X \cup \{\infty\}$  such that  $\infty \notin X$ . We consider:

$$\Sigma = (X, \mathcal{B}) = BCP^{(3)}(1, 5)\text{-design of order } 2h.$$

We now consider a factorization  $\mathcal{F}$  of the set  $X$ . Since the cardinality of  $X$  is even, we have  $2h - 1$  factors such that  $|F_i| = h$ , for all  $i = 1, \dots, 2h - 1$ :

$$\mathcal{F} = \{F_1, F_2, \dots, F_{2h-1}\}.$$

If the generic factor is  $F_i = \{(a_{i,1}, a_{i,2}); (a_{i,3}, a_{i,4}); \dots; (a_{i,2h-1}, a_{i,2h})\}$ , then for all  $i = 1, \dots, 2h - 1$  (i.e. for each factor) we construct the following blocks like that:

$$\Gamma_i = \{[a_{i,1}, a_{i,2}, (\infty), a_{i,3}, a_{i,4}]; \dots; [a_{i,2h-1}, a_{i,2h}, (\infty), a_{i,2}, a_{i,1}]; [a_{i,4}, a_{i,3}, (\infty), a_{i,6}, a_{i,5}]; \dots; [a_{i,2h-2}, a_{i,2h-3}, (\infty), a_{i,2h}, a_{i,2h-1}]\}.$$

Let  $\Gamma = \bigcup_{i=1}^{2h-1} \Gamma_i$ , then  $\Sigma' = (X', \mathcal{B}')$  is a  $BCP^{(3)}(1, 5)$ -design of order  $2h + 1$ , where  $X' = X \cup \{\infty\}$  and  $\mathcal{B}' = \mathcal{B} \cup \Gamma$ .

*Existence.* Let  $(x, y, z)$  be a cyclic triple of  $\Sigma'$ ; the following cases may occur:

1. if  $x, y, z \in X$ , then the triple is contained in exactly one block of  $\Sigma$ ;
2. if  $x, y \in X$  and  $z = \infty$ , then the triple is contained in at least one block of  $\Gamma$  since  $(x, y)$  will be in exactly one  $F_i \in \mathcal{F}$  of  $X$ ;
3. the remaining possible cases follow from the previous ones.

*Uniqueness.* First note that for  $v = 2h + 1$ , the number of blocks is given by:

$$|\mathcal{B}'| = \frac{v(v-1)(v-2)}{6} = \frac{(2h+1)2h(2h-1)}{6}.$$

Verifying, we obtain:

$$\begin{aligned} |\mathcal{B}'| &= |\mathcal{B}| + |\Gamma| = \frac{2h(2h-1)(2h-2)}{6} + \sum_{i=1}^{2h-1} |\Gamma_i| = \\ &= \frac{2h(2h-1)(2h-2)}{6} + (2h-1)h = \\ &= \frac{(2h-1)(4h^2+2h)}{6} = \frac{(2h+1)2h(2h-1)}{6}. \end{aligned}$$

**Case (2B):**  $v = 2h + 1, h \geq 3, h$  odd.

Let be  $X = \mathbb{Z}_{2h+1} = \{0, 1, \dots, 2h\}$  and  $X' = X \cup \{\infty\}$  such that  $\infty \notin X$ . We consider:

$$\Sigma = (X, \mathcal{B}) = BCP^{(3)}(1, 5)\text{-design of order } 2h + 1.$$

We now consider a pseudo-factorization  $\mathcal{F}$  of the set  $X$ . Since the cardinality of  $X$  is odd, we have  $2h + 1$  factors such that  $|F_i| = h$ , for all  $i = 1, \dots, 2h + 1$ :

$$\mathcal{F} = \{F_1, F_2, \dots, F_{2h+1}\}.$$

If the generic factor is  $F_i = \{(a_{i,1}, a_{i,2}); (a_{i,3}, a_{i,4}); \dots; (a_{i,2h}, a_{i,2h+1})\}$ , then for all  $i = 1, \dots, 2h + 1$  (i.e. for each factor) we construct the following blocks like that:

$$\begin{aligned} \Gamma_i &= \{[a_{i,1}, a_{i,2}, (\infty), a_{i,3}, a_{i,4}]; \dots; [a_{i,2h}, a_{i,2h+1}, (\infty), a_{i,2}, a_{i,1}]; \\ &\quad [a_{i,4}, a_{i,3}, (\infty), a_{i,6}, a_{i,5}]; \dots; [a_{i,2h-1}, a_{i,2h-2}, (\infty), a_{i,2h+1}, a_{i,2h}]\}. \end{aligned}$$

Let  $\Gamma = \bigcup_{i=1}^{2h+1} \Gamma_i$ , then  $\Sigma' = (X', \mathcal{B}')$  is a  $BCP^{(3)}(1, 5)$ -design of order  $2h + 2$ , where  $X' = X \cup \{\infty\}$  and  $\mathcal{B}' = \mathcal{B} \cup \Gamma$ .

*Existence.* Let  $(x, y, z)$  be a cyclic triple of  $\Sigma'$ ; the following cases may occur:

1. if  $x, y, z \in X$ , then the triple is contained in exactly one block of  $\Sigma$ ;
2. if  $x, y \in X$  and  $z = \infty$ , then the triple is contained in at least one block of  $\Gamma$  since  $(x, y)$  will be in exactly one  $F_i \in \mathcal{F}$  of  $X$ ;
3. the remaining possible cases follow from the previous ones.

*Uniqueness.* First note that for  $v = 2h + 2$ , the number of blocks is given by:

$$|\mathcal{B}'| = \frac{v(v-1)(v-2)}{6} = \frac{(2h+2)(2h+1)2h}{6}.$$

Verifying, we obtain:

$$\begin{aligned}
 |\mathcal{B}'| &= |\mathcal{B}| + |\Gamma| = \frac{(2h+1)2h(2h-1)}{6} + \sum_{i=1}^{2h+1} |\Gamma_i| = \\
 &= \frac{(2h+1)2h(2h-1)}{6} + (2h+1)h = \\
 &= \frac{(2h+1)(4h^2+4h)}{6} = \frac{(2h+2)(2h+1)2h}{6}.
 \end{aligned}$$

This completes the proof of the theorem on the spectrum of  $BCP^{(3)}(1, 5)$ -designs. The existence and structure of these designs are now fully determined.  $\square$

## 5. Spectrum of $BTP^{(3)}(2, 4)$ -designs and $BTP^{(3)}(1, 5)$ -designs

Let us first provide the definitions of these new structures in order to study their spectrum.

**Definition 5.1.** A block transitive  $P^{(3)}(2, 4)$  is a transitively oriented hypergraph  $(X, \mathcal{B})$  consisting of four vertices  $x, y, z, t$  and two transitive triples  $[y, z, x], [y, t, x]$ . Such a system will be denoted as  $BTP^{(3)}(2, 4) = [z, (y, x), t]$  and identifies the configuration in which the central oriented edge is determined by the ordered pair  $(y, x)$ .

**Definition 5.2.** A  $BTP^{(3)}(2, 4)$ -design of order  $v$  is a decomposition of the complete transitive hypergraph  $\mathcal{K}_v^{(3)}$  into hypergraphs isomorphic to  $BTP^{(3)}(2, 4)$ .

In order to study the spectrum, it is sufficient to observe that a  $BCP^{(3)}(2, 4)$  can always be transformed into a  $BTP^{(3)}(2, 4)$ . In fact, a cyclic triple can be made transitive by inverting the orientation of just one edge, that is, by modifying only one pair within the triple. In the case of a  $BCP^{(3)}(2, 4)$ , it is enough to invert the central edge: in this way, the triples become transitive, sharing one pair (the inverted one), thereby obtaining a  $BTP^{(3)}(2, 4)$ . Consequently, the same results established for  $BCP^{(3)}(2, 4)$ -designs also apply to  $BTP^{(3)}(2, 4)$ -designs.

**Theorem 5.3.** A  $BTP^{(3)}(2, 4)$ -design of order  $v$  exists for every  $v \geq 4$ .

**Definition 5.4.** A block transitive  $P^{(3)}(1, 5)$  is a transitively oriented hypergraph  $(X, \mathcal{B})$  consisting of five vertices  $x, y, z, t, w$  and two transitive triples  $[z, x, y], [w, x, t]$  that share the vertex  $x$  in common. Such a system will be denoted as  $BTP^{(3)}(1, 5) = [y, z, (x), t, w]$ , where the vertex  $x$  is the common (central) element in both triples.

**Definition 5.5.** A  $BTP^{(3)}(1, 5)$ -design of order  $v$  is a decomposition of the complete transitive hypergraph  $\mathcal{K}_v^{(3)}$  into hypergraphs isomorphic to  $BTP^{(3)}(1, 5)$ .

By the same reasoning, a  $BTP^{(3)}(1, 5)$  can be obtained from a  $BCP^{(3)}(1, 5)$ . In this case, the condition is less restrictive, since the two triples no longer share an edge, but only a single vertex. Consequently, it is sufficient to invert any two edges, one from each of the two triples contained in the  $BCP^{(3)}(1, 5)$ , to obtain a  $BTP^{(3)}(1, 5)$ . Therefore, also in this case, the same results established for  $BCP^{(3)}(1, 5)$ -designs apply to  $BTP^{(3)}(1, 5)$ -designs.

**Theorem 5.6.** A  $BTP^{(3)}(1, 5)$ -design of order  $v$  exists for every  $v \geq 5$ .

In conclusion, through the constructions presented, it was possible to determine the spectrum of four various  $\mathcal{H}^{(3)}$ -designs, thereby contributing to a broader understanding of the structure and existence of these configurations.

## 6. Appendix

### 6.1. Algorithm

The python algorithm presented below is designed to generate a  $BCP^{(3)}(2, 4)$ -design on an arbitrary number of vertices  $v \geq 4$ . Although its output is specifically tailored to this family of designs, the structure of the algorithm lends itself well to adaptations for constructing other types of designs.

In particular, with suitable modifications, the same framework can be adapted to construct a  $BCP^{(3)}(1, 5)$ -design. This requires both a different representation of the hypergraph (which in this case takes the form of a horizontally oriented hourglass) and a revision of the factors used in the constructions we require, which are useful for generating the triples. Moreover, based on what has been discussed previously, by appropriately inverting the order of the triples to ensure their transitivity, the approach can also be extended to the construction of  $BTP^{(3)}(2, 4)$ -design and  $BTP^{(3)}(1, 5)$ -design.

Hereafter, we will refer to the  $BCP^{(3)}(2, 4)$ -design on  $v$  vertices as a  $BCP(v)$ -design, or more briefly, as a  $BCP(v)$ .

#### $BCP(v)$ -design's construction - python code

```
import networkx as nx
import matplotlib.pyplot as plt
from itertools import combinations
from BCP_constr_2h_2h_plus_4 import generate_BCP_2h_2h_plus_4
from BCP_constr_2h_plus_1_2h_plus_5 import generate_BCP_2h_plus_1_2h_plus_5

def generate_BCP(v):

    if v < 4:
        raise ValueError("h must be greater than or equal to 2.")

    if v == 4:
        all_permutations = [
            [0, 1, 2, 3],
            [0, 2, 3, 1],
            [3, 1, 2, 0],
            [3, 2, 0, 1]
        ]

        print(f"total number of blocks for v = 4: {len(all_permutations)}")
        print(all_permutations)

    for idx, perm in enumerate(all_permutations):
        row = idx // 2
        col = idx % 2

        G = nx.DiGraph()
        G.add_edges_from([
            ('0', '1'),
            ('1', '2'),
            ('1', '3'),
            ('2', '0'),
            ('3', '0')
        ])

        base_positions = {
            '0': (0, 0),
            '1': (0, 2),
            '2': (-1, 1),
            '3': (1, 1)
        }

        shifted_positions = {}
```

```

        for node, (x, y) in base_positions.items():
            shifted_positions[node] = (x + col * 5, y - row * 5)

        labels = {node: label for node, label in zip(base_positions, perm)}

        nx.draw(G, shifted_positions, with_labels=True, labels=labels,
node_color='lightblue', edge_color='red', arrows=True)

        plt.show()

        return all_permutations

if v == 5:
    all_permutations = [
        [0, 2, 4, 1],
        [0, 3, 4, 1],
        [1, 3, 0, 2],
        [1, 4, 0, 2],
        [2, 4, 1, 3],
        [2, 0, 1, 3],
        [3, 0, 2, 4],
        [3, 1, 2, 4],
        [4, 1, 3, 0],
        [4, 2, 3, 0],
    ]

    print(f"total number of blocks for v = 5: {len(all_permutations)}")
    print(all_permutations)

    for idx, perm in enumerate(all_permutations):
        row = idx // 2
        col = idx % 2

        G = nx.DiGraph()
        G.add_edges_from([
            ('0', '1'),
            ('1', '2'),
            ('1', '3'),
            ('2', '0'),
            ('3', '0')
        ])

        base_positions = {
            '0': (0, 0),
            '1': (0, 2),
            '2': (-1, 1),
            '3': (1, 1)
        }

        shifted_positions = {}
        for node, (x, y) in base_positions.items():
            shifted_positions[node] = (x + col * 5, y - row * 5)

        labels = {node: label for node, label in zip(base_positions, perm)}

        nx.draw(G, shifted_positions, with_labels=True, labels=labels,
node_color='lightblue', edge_color='red', arrows=True)

        plt.show()

        return all_permutations

if v == 6:
    all_permutations = [
        [0, 2, 4, 1],
        [0, 3, 4, 1],

```

```

[1, 3, 0, 2],
[1, 4, 0, 2],
[2, 4, 1, 3],
[2, 0, 1, 3],
[3, 0, 2, 4],
[3, 1, 2, 4],
[4, 1, 3, 0],
[4, 2, 3, 0],
[0, 5, 1, 2],
[0, 5, 1, 3],
[0, 5, 1, 4],
[0, 5, 2, 3],
[0, 5, 2, 4],
[0, 5, 3, 4],
[1, 5, 2, 3],
[1, 5, 2, 4],
[1, 5, 3, 4],
[2, 5, 3, 4]
]

print(f"total number of blocks for v = 6: {len(all_permutations)}")
print(all_permutations)

for idx, perm in enumerate(all_permutations):
    row = idx // 5
    col = idx % 5

    G = nx.DiGraph()
    G.add_edges_from([
        ('0', '1'),
        ('1', '2'),
        ('1', '3'),
        ('2', '0'),
        ('3', '0')
    ])

    base_positions = {
        '0': (0, 0),
        '1': (0, 2),
        '2': (-1, 1),
        '3': (1, 1)
    }

    shifted_positions = {}
    for node, (x, y) in base_positions.items():
        shifted_positions[node] = (x + col * 6, y - row * 6)

    labels = {node: label for node, label in zip(base_positions, perm)}

    nx.draw(G, shifted_positions, with_labels=True, labels=labels,
node_color='lightblue', edge_color='red', arrows=True)

    plt.show()

    return all_permutations

if v == 7:
    all_permutations = [
        [0, 2, 4, 1],
        [0, 3, 4, 1],
        [1, 3, 0, 2],
        [1, 4, 0, 2],
        [2, 4, 1, 3],
        [2, 0, 1, 3],
        [3, 0, 2, 4],
        [3, 1, 2, 4],

```

```

[4, 1, 3, 0],
[4, 2, 3, 0],
[2, 5, 0, 1],
[4, 5, 0, 1],
[3, 5, 1, 2],
[0, 5, 1, 2],
[4, 5, 2, 3],
[1, 5, 2, 3],
[0, 5, 3, 4],
[2, 5, 3, 4],
[1, 5, 4, 0],
[3, 5, 4, 0],
[2, 6, 0, 1],
[3, 6, 0, 1],
[4, 6, 0, 1],
[4, 6, 2, 3],
[5, 6, 2, 3],
[0, 6, 2, 3],
[0, 6, 4, 5],
[1, 6, 4, 5],
[2, 6, 4, 5],
[6, 1, 0, 2],
[6, 2, 0, 1],
[6, 3, 2, 4],
[6, 4, 2, 3],
[6, 5, 4, 0],
[6, 0, 4, 5]
]

print(f"total number of blocks for v = 7: {len(all_permutations)}")
print(all_permutations)

for idx, perm in enumerate(all_permutations):
    row = idx // 5
    col = idx % 5

    G = nx.DiGraph()
    G.add_edges_from([
        ('0', '1'),
        ('1', '2'),
        ('1', '3'),
        ('2', '0'),
        ('3', '0')
    ])

    base_positions = {
        '0': (0, 0),
        '1': (0, 2),
        '2': (-1, 1),
        '3': (1, 1)
    }

    shifted_positions = {}
    for node, (x, y) in base_positions.items():
        shifted_positions[node] = (x + col * 6, y - row * 6)

    labels = {node: label for node, label in zip(base_positions, perm)}

    nx.draw(G, shifted_positions, with_labels=True, labels=labels,
node_color='lightblue', edge_color='red', arrows=True)

    plt.show()

    return all_permutations

if v > 7 and v % 2 == 0:

```

```

h = (v - 4) // 2
all_permutations = generate_BCP_2h_2h_plus_4(h)

print(f"total number of blocks for v = {v}: {len(all_permutations)}")
print(all_permutations)
generate_graphs(all_permutations)
return all_permutations

elif v > 7 and v % 2 == 1:
h = (v - 5) // 2
all_permutations = generate_BCP_2h_plus_1_2h_plus_5(h)

print(f"total number of blocks for v = {v}: {len(all_permutations)}")
print(all_permutations)
generate_graphs(all_permutations)
return all_permutations

def generate_graphs(permutations):
num_permutations = len(permutations)
graphs_per_row = 8
graphs_per_col = 7
graphs_per_grid = graphs_per_row * graphs_per_col

num_grids = (num_permutations + graphs_per_grid - 1) // graphs_per_grid

for grid_idx in range(num_grids):
plt.figure(figsize=(24, 21))

start_idx = grid_idx * graphs_per_grid
end_idx = min(start_idx + graphs_per_grid, num_permutations)

for idx, perm in enumerate(permutations[start_idx:end_idx]):
row = idx // graphs_per_row
col = idx % graphs_per_row

G = nx.DiGraph()
G.add_edges_from([
('0', '1'),
('1', '2'),
('1', '3'),
('2', '0'),
('3', '0')
])

base_positions = {
'0': (0, 0),
'1': (0, 2),
'2': (-1, 1),
'3': (1, 1)
}

shifted_positions = {}
for node, (x, y) in base_positions.items():
shifted_positions[node] = (x + col * 5, y - row * 5)

labels = {node: label for node, label in zip(base_positions, perm)}

nx.draw(G, shifted_positions, with_labels=True, labels=labels,
node_color='lightblue', edge_color='red', arrows=True)

plt.show()

def run_BCP():
v = int(input("insert the value of v (>= 4): "))
generate_BCP(v)

```

```
if __name__ == "__main__":
    run_BCP()
```

## 6.2. Examples

In the following, for the base cases  $v = 4, v = 5, v = 6, v = 7$  and for the first cases  $v = 8$  and  $v = 9$ , involved in the recursive constructions  $2h \rightarrow 2h + 4$  and  $2h + 1 \rightarrow 2h + 5$ , we show the outputs of the decomposition of the cyclic and inverse cyclic triples listed in the corresponding tables and figures below.

Table 1: Cyclic and inverse cyclic triples for  $v = 4$ .

<b>Cyclic triples</b>			
{0,1,2}	{0,1,3}	{0,2,3}	{1,2,3}
<b>Inverse cyclic triples</b>			
{1,0,2}	{1,0,3}	{2,0,3}	{2,1,3}

Figure 1: The  $BCP(4)$ -design is illustrated in the figure above.

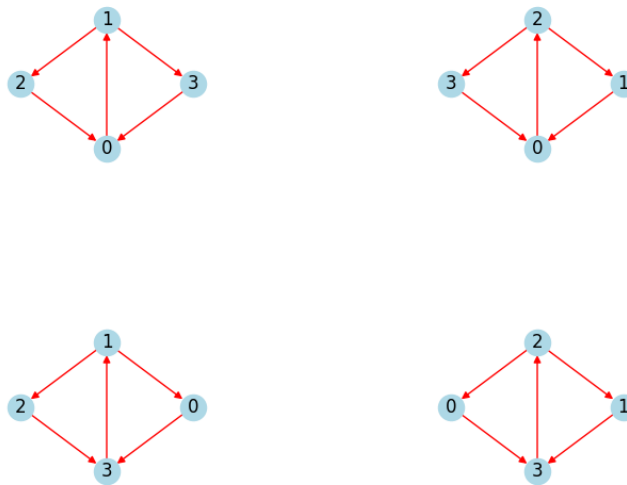


Table 2: Cyclic and inverse cyclic triples for  $v = 5$ .

Cyclic triples				
{0,1,2}	{0,1,3}	{0,1,4}	{0,2,3}	{0,2,4}
{0,3,4}	{1,2,3}	{1,2,4}	{1,3,4}	{2,3,4}

Inverse cyclic triples				
{1,0,2}	{1,0,3}	{1,0,4}	{2,0,3}	{2,0,4}
{3,0,4}	{2,1,3}	{2,1,4}	{3,1,4}	{3,2,4}

Figure 2: The  $BCP(5)$ -design is illustrated in the figure above.

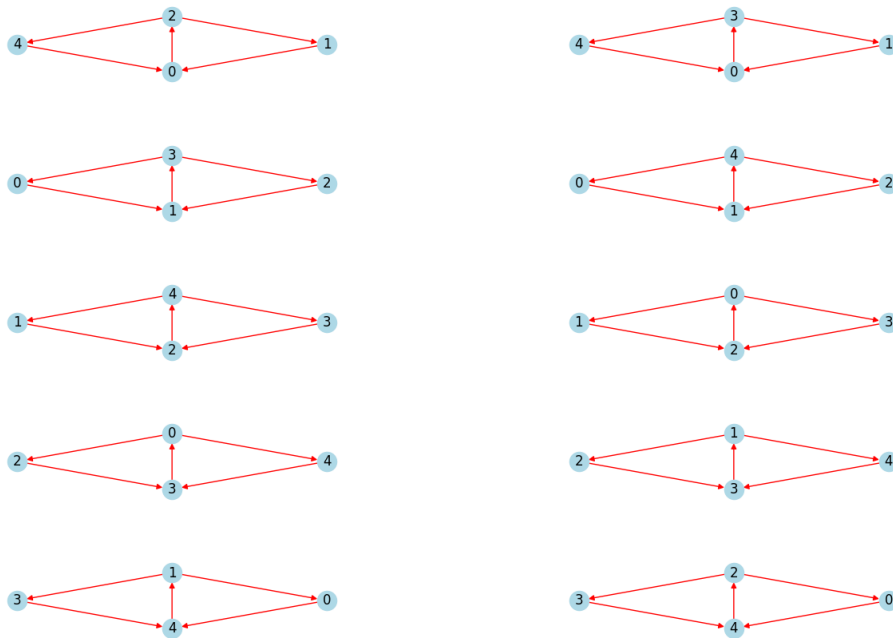


Table 3: Cyclic and inverse cyclic triples for  $v = 6$ .

Cyclic triples				
{0,1,2}	{0,1,3}	{0,1,4}	{0,1,5}	{0,2,3}
{0,2,4}	{0,2,5}	{0,3,4}	{0,3,5}	{0,4,5}
{1,2,3}	{1,2,4}	{1,2,5}	{1,3,4}	{1,3,5}
{1,4,5}	{2,3,4}	{2,3,5}	{2,4,5}	{3,4,5}

Inverse cyclic triples				
{1,0,2}	{1,0,3}	{1,0,4}	{1,0,5}	{2,0,3}
{2,0,4}	{2,0,5}	{3,0,4}	{3,0,5}	{4,0,5}
{2,1,3}	{2,1,4}	{2,1,5}	{3,1,4}	{3,1,5}
{4,1,5}	{3,2,4}	{3,2,5}	{4,2,5}	{4,3,5}

Figure 3: The  $BCP(6)$ -design is illustrated in the figure above.

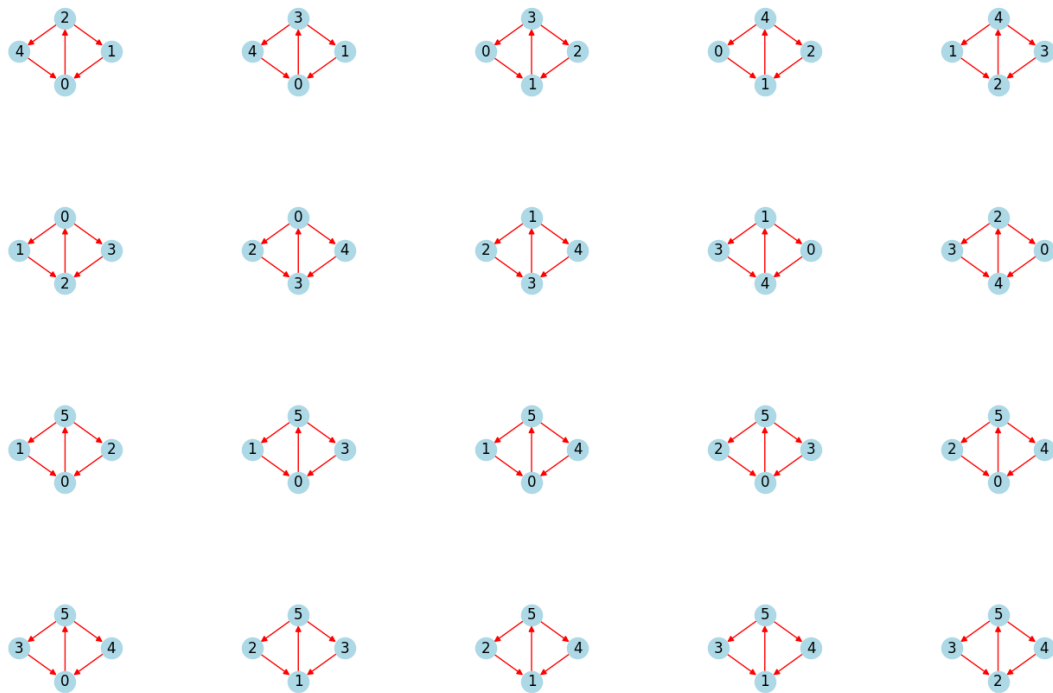


Table 4: Cyclic and inverse cyclic triples for  $v = 7$ .

Cyclic triples						
{0,1,2}	{0,1,3}	{0,1,4}	{0,1,5}	{0,1,6}	{0,2,3}	{0,2,4}
{0,2,5}	{0,2,6}	{0,3,4}	{0,3,5}	{0,3,6}	{0,4,5}	{0,4,6}
{0,5,6}	{1,2,3}	{1,2,4}	{1,2,5}	{1,2,6}	{1,3,4}	{1,3,5}
{1,3,6}	{1,4,5}	{1,4,6}	{1,5,6}	{2,3,4}	{2,3,5}	{2,3,6}
{2,4,5}	{2,4,6}	{2,5,6}	{3,4,5}	{3,4,6}	{3,5,6}	{4,5,6}

Inverse cyclic triples						
{1,0,2}	{1,0,3}	{1,0,4}	{1,0,5}	{1,0,6}	{2,0,3}	{2,0,4}
{2,0,5}	{2,0,6}	{3,0,4}	{3,0,5}	{3,0,6}	{4,0,5}	{4,0,6}
{5,0,6}	{2,1,3}	{2,1,4}	{2,1,5}	{2,1,6}	{3,1,4}	{3,1,5}
{3,1,6}	{4,1,5}	{4,1,6}	{5,1,6}	{3,2,4}	{3,2,5}	{3,2,6}
{4,2,5}	{4,2,6}	{5,2,6}	{4,3,5}	{4,3,6}	{5,3,6}	{5,4,6}

Figure 4: The  $BCP(7)$ -design is illustrated in the figure above.

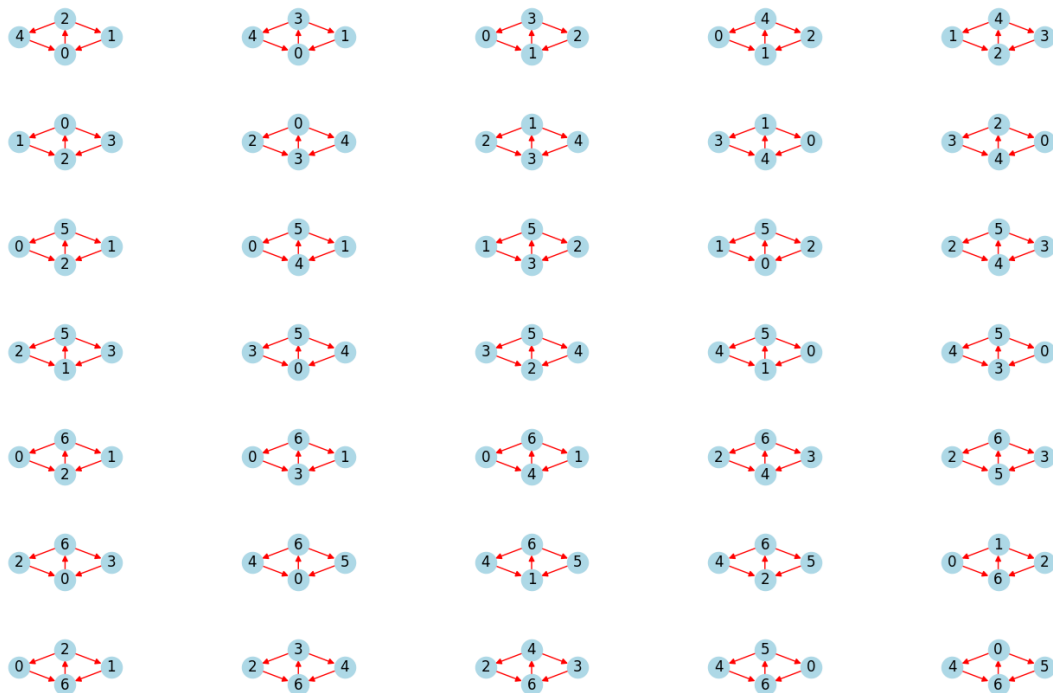


Table 5: Cyclic and inverse cyclic triples for  $v = 8$ .

**Cyclic triples**

{0,1,2}	{0,1,3}	{0,1,4}	{0,1,5}	{0,1,6}	{0,1,7}	{0,2,3}	{0,2,4}
{0,2,5}	{0,2,6}	{0,2,7}	{0,3,4}	{0,3,5}	{0,3,6}	{0,3,7}	{0,4,5}
{0,4,6}	{0,4,7}	{0,5,6}	{0,5,7}	{0,6,7}	{1,2,3}	{1,2,4}	{1,2,5}
{1,2,6}	{1,2,7}	{1,3,4}	{1,3,5}	{1,3,6}	{1,3,7}	{1,4,5}	{1,4,6}
{1,4,7}	{1,5,6}	{1,5,7}	{1,6,7}	{2,3,4}	{2,3,5}	{2,3,6}	{2,3,7}
{2,4,5}	{2,4,6}	{2,4,7}	{2,5,6}	{2,5,7}	{2,6,7}	{3,4,5}	{3,4,6}
{3,4,7}	{3,5,6}	{3,5,7}	{3,6,7}	{4,5,6}	{4,5,7}	{4,6,7}	{5,6,7}

**Inverse cyclic triples**

{1,0,2}	{1,0,3}	{1,0,4}	{1,0,5}	{1,0,6}	{1,0,7}	{2,0,3}	{2,0,4}
{2,0,5}	{2,0,6}	{2,0,7}	{3,0,4}	{3,0,5}	{3,0,6}	{3,0,7}	{4,0,5}
{4,0,6}	{4,0,7}	{5,0,6}	{5,0,7}	{6,0,7}	{2,1,3}	{2,1,4}	{2,1,5}
{2,1,6}	{2,1,7}	{3,1,4}	{3,1,5}	{3,1,6}	{3,1,7}	{4,1,5}	{4,1,6}
{4,1,7}	{5,1,6}	{5,1,7}	{6,1,7}	{3,2,4}	{3,2,5}	{3,2,6}	{3,2,7}
{4,2,5}	{4,2,6}	{4,2,7}	{5,2,6}	{5,2,7}	{6,2,7}	{4,3,5}	{4,3,6}
{4,3,7}	{5,3,6}	{5,3,7}	{6,3,7}	{5,4,6}	{5,4,7}	{6,4,7}	{6,5,7}

Figure 5: The  $BCP(8)$ -design is illustrated in the figure above.

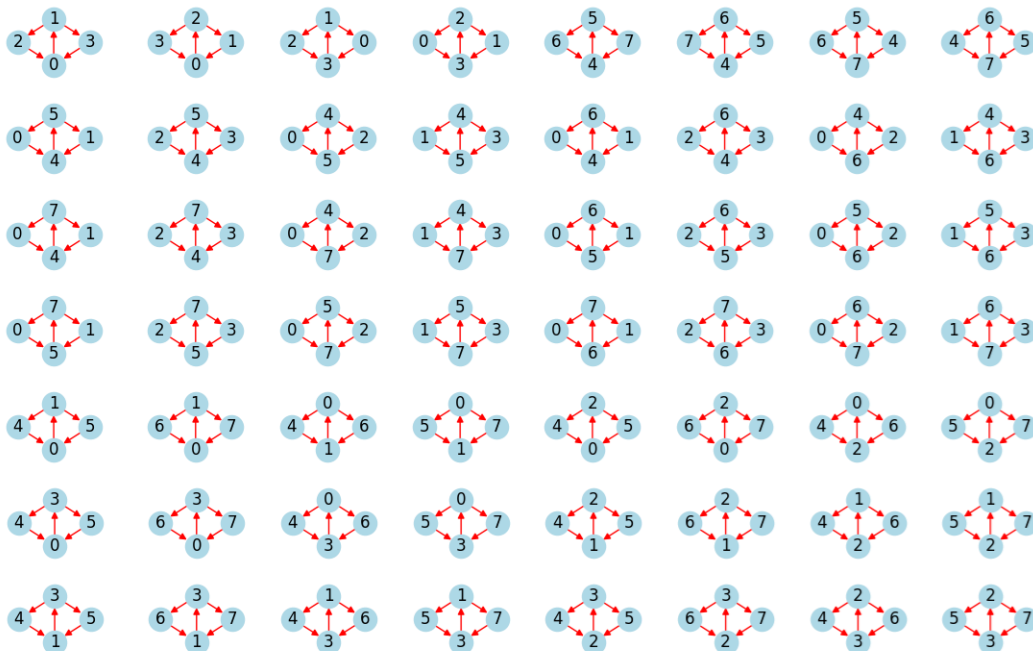
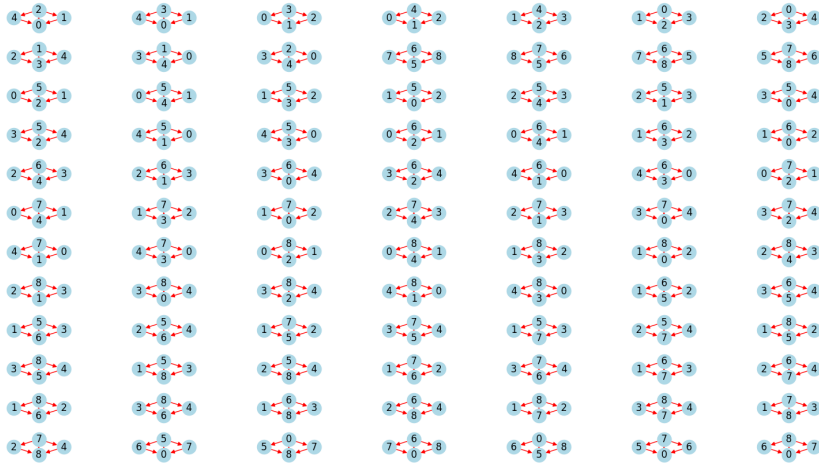


Table 6: Cyclic and inverse cyclic triples for  $v = 9$ .

Cyclic triples											
{0,1,2}	{0,1,3}	{0,1,4}	{0,1,5}	{0,1,6}	{0,1,7}	{0,1,8}	{0,2,3}	{0,2,4}	{0,2,5}	{0,2,6}	{0,2,7}
{0,2,8}	{0,3,4}	{0,3,5}	{0,3,6}	{0,3,7}	{0,3,8}	{0,4,5}	{0,4,6}	{0,4,7}	{0,4,8}	{0,5,6}	{0,5,7}
{0,5,8}	{0,6,7}	{0,6,8}	{0,7,8}	{1,2,3}	{1,2,4}	{1,2,5}	{1,2,6}	{1,2,7}	{1,2,8}	{1,3,4}	{1,3,5}
{1,3,6}	{1,3,7}	{1,3,8}	{1,4,5}	{1,4,6}	{1,4,7}	{1,4,8}	{1,5,6}	{1,5,7}	{1,5,8}	{1,6,7}	{1,6,8}
{1,7,8}	{2,3,4}	{2,3,5}	{2,3,6}	{2,3,7}	{2,3,8}	{2,4,5}	{2,4,6}	{2,4,7}	{2,4,8}	{2,5,6}	{2,5,7}
{2,5,8}	{2,6,7}	{2,6,8}	{2,7,8}	{3,4,5}	{3,4,6}	{3,4,7}	{3,4,8}	{3,5,6}	{3,5,7}	{3,5,8}	{3,6,7}
{3,6,8}	{3,7,8}	{4,5,6}	{4,5,7}	{4,5,8}	{4,6,7}	{4,6,8}	{4,7,8}	{5,6,7}	{5,6,8}	{5,7,8}	{6,7,8}
Inverse cyclic triples											
{1,0,2}	{1,0,3}	{1,0,4}	{1,0,5}	{1,0,6}	{1,0,7}	{1,0,8}	{2,0,3}	{2,0,4}	{2,0,5}	{2,0,6}	{2,0,7}
{2,0,8}	{3,0,4}	{3,0,5}	{3,0,6}	{3,0,7}	{3,0,8}	{4,0,5}	{4,0,6}	{4,0,7}	{4,0,8}	{5,0,6}	{5,0,7}
{5,0,8}	{6,0,7}	{6,0,8}	{7,0,8}	{2,1,3}	{2,1,4}	{2,1,5}	{2,1,6}	{2,1,7}	{2,1,8}	{3,1,4}	{3,1,5}
{3,1,6}	{3,1,7}	{3,1,8}	{4,1,5}	{4,1,6}	{4,1,7}	{4,1,8}	{5,1,6}	{5,1,7}	{5,1,8}	{6,1,7}	{6,1,8}
{7,1,8}	{3,2,4}	{3,2,5}	{3,2,6}	{3,2,7}	{3,2,8}	{4,2,5}	{4,2,6}	{4,2,7}	{4,2,8}	{5,2,6}	{5,2,7}
{5,2,8}	{6,2,7}	{6,2,8}	{7,2,8}	{4,3,5}	{4,3,6}	{4,3,7}	{4,3,8}	{5,3,6}	{5,3,7}	{5,3,8}	{6,3,7}
{6,3,8}	{7,3,8}	{5,4,6}	{5,4,7}	{5,4,8}	{6,4,7}	{6,4,8}	{7,4,8}	{6,5,7}	{6,5,8}	{7,5,8}	{7,6,8}

Figure 6: The  $BCP(9)$ -design is illustrated in the figure above.



**Acknowledgments:** The authors thank the referees for their useful comments.

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