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A note on the algebra of threshold graphs

Research Article

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Abstract: It is known that threshold graphs have edge rings with 2-linear resolutions. This was proved by Engström and Stamps, [4]. They used the fact that an edge ring of a graph G has a 2-linear resolution if and only if the complement graph is chordal. They also described a method to determine the Betti numbers. Our goal is to determine when edge rings of threshold graphs are Cohen-Macaulay. In order to do so, it is more convenient to use an alternative way to study edge rings of graphs, that is to interpret them as Stanley-Reisner rings. We also determine when the neighborhood complex of a threshold graph has a Cohen-Macaulay Stanley-Reisner ring.

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1. Introduction

A simplicial complex Δ consists of a set of subsets (faces) σ of a finite set $V = \{v_1, \ldots, v_n\}$, such that $\{v_i\} \in \Delta$ for all i, and if $\sigma_1 \in \Delta$ and $\sigma_2 \subset \sigma_1$, then $\sigma_2 \in \Delta$. Maximal faces are called facets. If Δ is a simplicial complex with vertices $\{v_1, \ldots, v_n\}$, its Stanley-Reisner ring over a field k is $k[\Delta] = k[x_1, \ldots, x_n]/I_\Delta$, where its Stanley-Reisner ideal I_Δ is generated by the squarefree monomials $x_{i_1} \cdots x_{i_k}$ for all non-faces $\{v_{i_1}, \ldots, v_{i_k}\}$ of Δ. If Δ is a simplicial complex on $\{1, 2, \ldots, n\} = [1, n]$, then the Alexander dual $\Delta^\vee$ is the complex on $[1, n]$ for which $\sigma \subseteq [1, n]$ is a face of $\Delta^\vee$ if and only if $[1, n] \setminus \sigma$ is not a face of Δ. The f-vector of a simplicial complex Δ on $[1, n]$ is $f(\Delta) = (f_{-1}, f_0, \ldots, f_{n-1})$, where f_i is the number of i-dimensional faces in Δ. Let $f_{-1} = 1$. We usually skip the zeros at the end.

A graph $G = (V, E)$ is a finite set of vertices $V = \{v_1, \ldots, v_n\}$ and edges $E = \{(v_i, v_j)\}$. A vertex v is called isolated if v is not contained in any edge. A subset $W \subseteq V$ is called independent if the graph induced on W consists of isolated vertices. Two vertices v and w are called neighbors if (v, w) is an edge. For a vertex v, let $N(v)$ consist of all neighbors to v. The edge ring of $G = (V, E)$ is $k[G] = k[x_1, \ldots, x_n]/I(G)$, where $V = \{v_1, \ldots, v_n\}$ and the edge ideal $I(G)$ is generated by $\{x_i x_j; (v_i, v_j) \in E\}$. The complement graph to $G = (V, E)$ is $\bar{G} = (V, \overline{E})$, where $\overline{E} = \{(x_i, x_j); (i, j) \notin E\}$.

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A graded algebra $R = k[x_1, \ldots, x_n]/I = S/I = \bigoplus_{i \geq 0} R_i$ has Hilbert series $R(t) = \sum_{i \geq 0} \dim_k R_i t^i$. It has a t-linear resolution if I is generated in degree t and all higher syzygies are linear, or equivalently, if $\operatorname{Tor}^S_{i,j}(R, k) = 0$ if $j \neq i + t - 1$. A t-linear algebra R has Hilbert series $R(t) = (1 + \sum_{i=1}^n (-1)^i \beta_{i,i+t-1}(R) t^{i+t-1})/(1-t)^n$, where $\beta_{i,j}(R) = \dim_k \operatorname{Tor}^S_{i,j}(R, k)$. Thus, if R has a t-linear resolution, the Hilbert series contains the same information as the set of all Betti numbers. We will concentrate on 2-linear resolutions and denote the vector $(\beta_{1,2}(R), \beta_{2,3}(R), \ldots, \beta_{n,n+1}(R))$ by $\beta(R)$. We skip zeros at the end.

We now define some graph complexes, i.e., simplicial complexes belonging to a graph.

Definition 1.1. *For a graph $G = (V, E)$, the neighborhood complex $N(G)$ consists of all subsets of $N(v)$ for all $v \in V$.*

The dominance complex $D(G)$ of a graph $G = (V, E)$ consists of those $W \subseteq V$ such that $V \setminus W$ is dominating. A subset U of V is dominating if every vertex in V either belongs to U or has a neighbor in U.

The independence complex $Ind(G)$ of a graph $G = (V, E)$ consists of the independent subsets of V. The Stanley-Reisner ideal $I_{Ind(G)}$ equals the edge ideal $I(G)$.

The definition of threshold graphs, which also explains the name, stems from [1].

Definition 1.2. *(Chvátal and Hammer, [1]) A graph $G = (V, E)$ is called a threshold graph when there exist nonnegative reals (or integers) w_v and t such that $\sum_{v \in U} w_v \leq t$ if and only if U is an independent subset of V.*

Theorem 1.3. *The following conditions are equivalent for a graph G:*

1. *G is a threshold graph.*
2. *G is a split graph, i.e., V can be partioned in a clique and an independent set, and the neighborhoods of vertices in the independent set are nested.*
3. *G can be constructed from a vertex by repeatedly adding an isolated vertex or a star vertex (i.e. with edges to all previous vertices).*
4. *The dominance complex $D(G)$ of G equals the neighborhood complex $N(G)$ of G.*

Proof. The equivalence of 1., 2. and 3. is proved in [8]. The equivalence of 1. and 4. is in [7]. □

There are more equivalent conditions proved in [8] and [7]. Here we add a further equivalent condition.

Theorem 1.4. *The Alexander dual to the neighborhood complex of G equals the neighborhood complex of the complement to G, that is $N(G)^\vee = N(\overline{G})$, if and only if G is threshold.*

Proof. That for any G we have $D(G) = N(\overline{G})^\vee$ is proved in [9]. Thus according to 4., we get that 1. is equivalent to $N(\bar{G})^\vee = N(G)$, and so is $N(\bar{G}) = N(G)^\vee$. □

Theorem 1.5. *[5] For a graph $G = ([1, n], E)$, $k[x_1, \ldots, x_n]/I(G)$ has a 2-linear resolution if and only if $\overline{G}$ is a chordal graph, i.e., every cycle in $\overline{G}$ of length > 3 has a chord.*

This theorem is made more precise in [2]. If $W \subset V$ let $G[W]$ be the graph induced on W.

Theorem 1.6. *[2, Theorem 3.2] Let $\overline{G}$ be a chordal graph. If $i \neq j - 1$ then $\beta_{i,j}(k[G]) = 0$ and otherwise $\beta_{i,j}(k[G]) = \sum_{W \in \binom{V}{j}} (-1 + \#$ connected components of $\overline{G[W]})$.*

It is also known which simplicial complexes that have a Stanley-Reisner ring with 2-linear resolution. These are the so called fat forests. Here is a recursive definition of fat forests, [5], [6]. A simplex F_1 is a fat forest. If F_i, $i = 1, \dots, k$ are simplices and $G_{k-1} = \cup_{I=1}^{k-1} F_i$ is a fat forest, then $G_{k-1} \cup F_k$ is a fat forest if $H = G_{k-1} \cap F_k$ is a simplex, $\dim H \geq -1$. (If $\dim H = -1$, then G_{k-1} and F_k are disjoint.) As an example, let $F_1, \dots, F_4$ be the simplices on $\{1,2,3,4\}, \{1,2,3,5\}, \{1,2,6\}, \{5,7\}$ respectively. The intersections are the simplices on $\{1,2,3\}, \{1,2\}$, and $\{5\}$, respectively. Then we build the fat forest with vertices $\{1,2,3,4,5,6,7\}$ and maximal faces F_1, F_2, F_3, F_4.

Theorem 1.7. *[5, 6] $k[\Delta]$ has a 2-linear resolution if and only if Δ is a fat forest.*

The projective dimension of a ring $R = S/I$ is $\max\{i; \mathrm{Tor}_{i,j}^S \neq 0\}$. The regularity of R is $\max\{j - i; \mathrm{Tor}_{i,j}^S \neq 0\}$. The depth of R is the maximal length of a regular sequence in R. Finally, R is Cohen-Macaulay if $\mathrm{depth}(R) = \dim(R)$. It is easy to get the Hilbert series, the dimension, and the depth for the Stanley-Reisner ring of a fat forest.

Theorem 1.8. *[6, Theorem 1] Let $F = F_1 \cup \dots \cup F_k$ be a fat forest with F_i a simplex of dimension d_i and $F_1 \cup \dots \cup F_{j-1} \cap F_j$ a simplex of dimension r_j. Then the Hilbert series of $k[F]$ is*

$$\sum_{i=1}^{k} \frac{1}{(1-t)^{d_i+1}} - \sum_{i=2}^{k} \frac{1}{(1-t)^{r_i+1}}.$$

The dimension of $k[G]$ is $\max\{d_i+1\}_{i=1}^k$ and the depth of $k[G]$ is $\min\{r_i+1\}_{i=1}^k$. $k[G]$ is Cohen-Macaulay if and only if there is a d such that $d_i = r_i + 1 = d$ for all i.

Theorem 1.9. *[3, Theorem 3][10, 11] $k[\Sigma]$ has a linear resolution if and only if $k[\Sigma^\vee]$ is Cohen-Macaulay. The regularity of I_Σ equals the projective dimension of $k[\Sigma^\vee]$, and vice versa.*

2. Threshold graphs

We will use 3. in Theorem 1.7 to study threshold graphs.. We will denote a graph constructed in this way by $G = I_{1,1} \cdots I_{1,i_1} S_1 I_{2,1} \cdots I_{2,i_2} \cdots S_2 I_{3,1} \cdots I_{3,i_3} \cdots S_{k-1} I_{k,1} \cdots I_{k,i_k}$, where the I's stand for adding an isolated vertex, and the S's for adding a star vertex. We call this a presentation of G. We demand that $i_1 > 0$ to get uniqueness (one can start with a I or with an S), but all other i_j may be 0. To adjoin an I at the end corresponds to adding a new variable, which does not affect the Betti numbers, so when we are interested in Betti numbers, we can suppose that the presentation ends with an S. It is not hard to see that the maximal independent sets of G are $I_{1,1} \cdots I_{k,i_k}$ (all I's) and all $S_i I_{i+1,1} \cdots I_{k,j_k}$ (each S followed by all I's which come after S).

In a very interesting paper [4], Engström and Stamps consider Boij-Söderberg decomposition of rings with 2-linear resolution. Among other things they show that $k[G]$ has a 2-linear resolution if G is a threshold graph by using Theorem 1.6. They also show how the Betti numbers change when adding an I or an S, thereby giving a method to determine the Betti numbers for any threshold graph. For our purpose it is easier to use Theorem 1.7 to show 2-linearity and Theorem 1.8 to get the Hilbert series, and thus the Betti numbers. We give an example.

Example 2.1. *Consider the threshold graph G with presentation $I_1 S_2 S_3 I_4 I_5 S_6$. The maximal independent sets are $\{I_1, I_4, I_5\}$, $\{S_2, I_4, I_5\}$, $\{S_3, I_4, I_5\}$, and $\{S_6\}$. We start with $\{I_1, I_4, I_5\}$ with series $1/(1-t)^3$ and attach $\{S_2, I_4, I_5\}$ with series $1/(1-t)^3$ in $\{I_4, I_5\}$ with series $1/(1-t)^2$. So far we get the series $2/(1-t)^3 - 1/(1-t)^2$. Then we attach $\{S_3, I_4, I_5\}$ with series $1/(1-t)^3$ in $\{I_4, I_5\}$ with series $1/(1-t)^2$ and get the series $3/(1-t)^3 - 2/(1-t)2$. Finally we attach $\{S_6\}$ with series $1/(1-t)$ in the empty set with series 1. The final result is $k[G](t) = 3/(1-t)^3 - 2/(1-t)^2 + 1/(1-t) - 1 = (1 - 8t^2 + 15t^3 - 12t^4 + 5t^5 - t^6)/(1-t)^6$. Thus $\beta(k[G]) = (8, 15, 12, 5, 1)$.*

The following picture illustrates the graph of $I_1 S_2 S_3 I_4 I_5 S_6$.

In the following theorem and corollaries we suppose that we have threshold graphs.

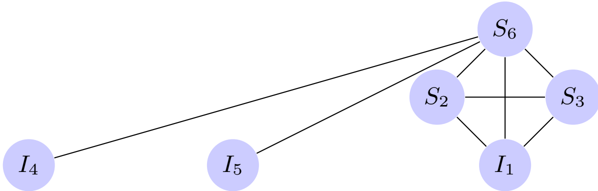

Figure 1. Graphs of $I_1S_2S_3I_4I_5S_6$.

Theorem 2.2. *The dimension of $k[G]$, equals the number of I's in the presentation of G. Suppose that the presentation ends with an S. Then the depth of $k[G]$ is always 1. Both the dimension and the depth increases by one for each I we have after the last S in the presentation.*

Proof. We use Theorem 1.8. We have that $\max\{d_i + 1\}$ equals the number of I's in the presentation of G, since the number of I's in the presentation is at least one more than the number of I's that follow an S, since the presentation starts with an I. Now $\min\{r_i + 1\} = 1$ if the presentation ends with an S. Adding an I at the end corresponds to adjoining a new variable which increases both dimension and depth by one. □

Corollary 2.3. *We have that $k[G]$ is Cohen-Macaulay if and only if it has a presentation consisting of only I's or is of the form I followed by a number of S's, and possibly followed by a number of I's. Thus $k[G]$ is Cohen-Macaulay if and only if G is either a graph of isolated vertices, or a complete graph and a (possibly empty) set of isolated vertices.*

Proof. If the presentations consists of only I's, we have a polynomial ring. If not, we can suppose the presentation ends with an S by above. Then $\dim(k[G]) = \operatorname{depth}(k[G]) = 1$ if and only if G has presentation $ISS\cdots S$. □

Corollary 2.4. *Suppose $k[G]$ has presentation $I_1S_2S_3\cdots S_n$. The Hilbert series of $k[G]$ is $(1+(n-1)t)/(1-t)$. In this case $k[G]^\vee$, the Alexander dual of $k[G]$, has an $(n-1)$-linear resolution with nonzero Betti numbers $\beta_{0,0} = 1$, $\beta_{1,n-1} = n$, $\beta_{2,n} = n-1$ and Hilbert series*

$$(1 - nt^{n-1} + (n-1)t^n)/(1-t)^n = (1 + 2t + 3t^2 + 4t^3 + \cdots + (n-1)t^{n-2})/(1-t)^{n-2}.$$

Proof. $I(G)$ has regularity 2 and $k[G]$ has projective dimension $n-1$, so $I(G^\vee)$ has regularity $n-1$ and $k[G]^\vee$ has projective dimension 2 according to Theorem 1.9. In case $k[G]$ is Cohen-Macaulay, $I(G^\vee)$ is generated by all n squarefree monomials of degree $n-1$, so for $\beta_{i,j}(k[G]^\vee)$ we get $\beta_{1,n-1} = n$. Since $\beta_{0,0} = 1$, the resolution is linear, and the alternating sum $\beta_{0,0} - \beta_{1,n-1} + \beta_{2,n} = 0$, we get $\beta_{2,n} = n-1$, and these are the only nonzero Betti numbers.The Hilbert series is thus $(1 - nt^{n-1} + (n-1)t^n)/(1-t)^n$. □

We end this section with a list over threshold graphs with at most 5 vertices and presentation ending with an S.

Table 1. Threshold graphs with at most five vertices.

Type	β	dim	depth	f-vector
IS	(1)	1	1	(1,2)
IIS	(2,1)	2	1	(1,3,1)
ISS	(3,2)	1	1	(1,3)
IIIS	(3,3,1)	3	1	(1,4,3,1)
ISIS	(4,4,1)	2	1	(1,4,2)
IISS	(5,6,2)	2	1	(1,4,1)
ISSS	(6,8,3)	1	1	(1,4)
IIIIS	(4,6,4,1	4	1	(1,5,6,4,1)
ISIIS	(5,7,4,1)	3	1	(1,5,6,2)
IISIS	(6,9,5,1)	3	1	(1,5,4,1)
IIISS	(7,12,8,2)	3	1	(1,5,3,1)
ISSIS	(7,11,6,1)	2	1	(1,5,3)
ISISS	(8,14,9,2)	2	1	(1,5,2)
IISSS	(9,17,12,3)	2	1	(1,5,1)
ISSSS	(10,20,15,4)	1	1	(1,5)

3. Neighborhood complexes for threshold graphs

Now we will determine which threshold graphs have Cohen-Macaulay neighborhood (or equivalently dominance) complexes.

Theorem 3.1. *The threshold graph G has a Cohen-Macaulay neighborhood complex, i.e., $k[N(G)]$ is Cohen-Macaulay, if and only if $k[G]$ is Cohen-Macaulay. In this case $k[N(G)]$ is a hyperplane section. Furthermore this occurs precisely when all facets in $N(G)$ have the same dimension.*

Proof. We can suppose that the presentation ends with an S, because I's at the end have no neighbors, so they only contribute with new variables. If the presentation is $I_1 \cdots S_n$, then S_n has $n-1$ neighbors, the maximal possible number. In the Stanley-Reisner ring $k[\Sigma]$ which is Cohen-Macaulay, all facets in Σ have the same dimension. Thus I_1 must have $n-1$ neighbors if $N(G)$ is Cohen-Macaulay. This gives that there are only S's after I_1. In this case $N(G)$ is the $(n-2)$-dimensional skeleton of the $(n-1)$-dimensional simplex on $\{I_1, S_2, \ldots, S_n\}$, which gives $k[N(G)] = k[x_1, \ldots, x_n]/(x_1 x_2 \cdots x_n)$. □

We finish by giving a list of Hilbert series, dimension, depth of $k[N(G)]$ and f-vector of $N(G)$ for all threshold graphs with at most five vertices and presentation ending with an S.

Table 2. Hilbert series, dimension, depth of $k[N(G)]$ and f-vector of $N(G)$ for all threshold graphs.

Type	$k[N(G)](t)$	dim	depth	f-vector
IS	$(1+t)/(1-t)$	1	1	(1,2)
IIS	$(1+t-t^2)/(1-t)^2$	2	1	(1,3,1)
ISS	$(1+t+t^2)/(1-t)^2$	2	2	(1,3,3)
IIIS	$(1+t-2t^2+t^3)/(1-t)^3$	3	1	(1,4,3,1)
ISIS	$(1+t-t^3)/(1-t)^3$	3	2	(1,4,5,1)
IISS	$(1+t+t^2-t^3)/(1-t)^3$	3	2	(1,4,6,2)
ISSS	$(1+t+t^2+t^3)/(1-t)^3$	3	3	(1,4,6,4)
IIIIS	$(1+t-3t^2+3t^3-t^4)/(1-t)^4$	4	1	(1,5,6,4,1)
ISIIS	$(1+t-t^2-t^3+t^4)/(1-t)^4$	4	2	(1,5,8,4,1)
IISIS	$(1+t-2t^3+t^4)/(1-t)^3$	4	2	(1,5,9,5,1)
IIISS	$(1+t+t^2-2t^3+t^4)/(1-t)^4$	4	2	(1,5,10,7,2)
ISSIS	$(1+t-t^4)/(1-t)^4$	4	3	(1,5,9,7,1)
ISISS	$(1+t+t^2-t^4)/(1-t)^4$	4	3	(1,5,10,9,2)
IISSS	$(1+t+t^2+t^3-t^4)/(1-t)^4$	4	3	(1,5,10,9,3)
ISSSS	$(1+t+t^2+t^3)/(1-t)^4$	4	4	(1,5,10,10,5)

4. Conclusion

Edge rings of threshold graphs were shown to have 2-linear resolution by Engström and Stamps. They used that threshold graphs are co-cordal. We determined when they are Cohen-Macaulay by instead showing that they have Stanley-Reisner rings which are fat forests. We also determined when the neighborhood complex of a threshold graph is Cohen-Macaulay.

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