J. Algebra Comb. Discrete Appl.
13(2) • 217–224

Received: 26 July 2025
Accepted: 15 March 2026

Journal of Algebra Combinatorics Discrete Structures and Applications

Expansion of implicit functions into formal power series in terms of partial Bell polynomials

Research Article

Alfred Schreiber

Abstract: Starting from the representation of a function $f(x, y)$ as a formal power series with Taylor coefficients $f_{m,n}$, a formal series is set up for the implicit function $y = y(x)$ so that $f(x, y) = 0$ and the coefficients of the series for y depend exclusively on the $f_{m,n}$. The solution to this problem provided here relies on using partial Bell polynomials and their inverse companions. Some examples and applications are discussed.

2020 MSC: 13F25, 11B83, 05A19, 11C08

Keywords: Implicit function, Formal power series, Higher derivatives, Series reversion, Bell polynomials, Multivariable Stirling polynomials

1. Introduction

The problem of calculating the higher derivatives of a function $y = y(x)$, which is implicitly given by an equation $f(x, y) = 0$, has already been dealt with several times in the mathematical literature of the 19th and 20th centuries. L. Comtet has listed some of these papers in the bibliography of his famous monograph [2]. His own contribution to the problem can be found in [1–3]. Recently, the problem has attracted renewed attention, especially with regard to some of its combinatorial aspects. The results in [3] have been subjected to careful analysis by Wilde [11], who also gives new proofs. Johnson [4] presents an elementary approach which leads to a recursive solution. Zemel [12] provides an in-depth combinatorial interpretation for those binomial building blocks that appear in the closed formula he proved for the higher derivatives of y. In the approach presented here, the combinatorial structure of the coefficients of y is essentially described by partial exponential polynomials (similar to [1]), which leads to less complicated formulas. Furthermore, an alternative form of representation was achieved by a modified type of series reversion (Section 3). For the sake of clarity, the entire procedure has been divided into two reduction steps (Section 4 and Section 6).

Alfred Schreiber (Corresponding Author); Department of Mathematics and Mathematical Education, Europa-Universität Flensburg, EUF, Auf dem Campus 1, 24943 Flensburg, Germany (email:agt.schreiber@proton.me).

https://doi.org/10.13069/jacodesmath.v13i2.397

ISSN 2148-838X

2. Comtet's problem

The procedure described in the following for calculating the higher derivatives of an implicit function starts from the problem as formulated by Comtet in [2, p. 152–153]. There, for a function f given as a formal power series

$$f(x,y)=\sum_{m,n\geq 0} f_{m,n}\frac{x^m y^n}{m!n!} \tag{1}$$

(with coefficients $f_{m,n}$ from a fixed commutative field of characteristic zero) Comtet poses the task of finding a formal power series $y=y(x)=\sum_{n\geq 1} y_n\frac{x^n}{n!}$ such that $f(x,y)=0$. We write D for the usual differential operator $\frac{d}{dx}$. In order to be able to compute the Taylor coefficients $y_n=D^n(y)(0)$, we assume $f_{0,0}=0$ and $f_{0,1}\neq 0$. Then, by writing

$$f(x,y)=\sum_{n\geq 0}\varphi_n\frac{y^n}{n!},$$

where $\varphi_n=\varphi_n(x):=\sum_{m\geq 0} f_{m,n}\frac{x^m}{m!}$, we see that $f(x,y)=0$ is equivalent to

$$g(y):=\sum_{n\geq 1}\varphi_n(x)\frac{y^n}{n!}=-\varphi_0(x)=-\left(f_{1,0}x+f_{2,0}\frac{x^2}{2!}+f_{3,0}\frac{x^3}{3!}+\cdots\right) \tag{2}$$

This formal power series g is invertible (with respect to the composition $\circ$), since we have $g(0)=0$ and because of $f_{0,1}\neq 0$ also $\frac{\partial g}{\partial y}(0)=\varphi_1(x)=f_{0,1}+x\cdot\sum(\ldots)\neq 0$. Let $\overline{g}$ denote the (unique) inverse of g. Then, the implicit function y is obtained from (2) in the form

$$y=y(x)=\overline{g}(-\varphi_0(x)). \tag{3}$$

In [2] Comtet evaluates this expression using the Lagrange inversion formula and finds the coefficients y_n by collecting the terms in $x^n/n!$ that occur in this process. Some few *ad hoc* calculations are performed that yield explicit formulas for y_1, y_2, y_3, y_4 (see the table on p. 153). However, this does not tell us what the general coefficient y_n actually looks like.

3. Preliminaries on Stirling polynomials

In his preceding work [1], Comtet had already developed an approach to represent y_n with the help of Bell polynomials and thus provide a better insight into the highly complex structure of y_n as a function of the coefficients of $f(x,y)$. In the following, we take up this idea and carry it out with a modified method that uses the *inverse companions* $A_{n,k}$ of the partial Bell (or exponential) polynomials $B_{n,k}$, which Comtet did not know at his time.

First, recall that $B_{n,k}$ can be introduced by the following equation:

$$\frac{1}{k!}\psi(x)^k=\sum_{n\geq k}B_{n,k}(\psi_1,\ldots,\psi_{n-k+1})\frac{x^n}{n!}, \tag{4}$$

where $\psi_1,\psi_2,\psi_3,\ldots$ are the Taylor coefficients of the function (formal power series) $\psi(x)$ (see, for instance, [2, p. 133]). The $B_{n,k}$ thus turn out to be polynomials in the indeterminates $X_1,\ldots,X_{n-k+1}$. As is well known (cf. [1, eq. (2)]), they can be explicitly expressed as the sum of monomial terms:

$$B_{n,k}=\sum_{\mathbb{P}(n,k)}\frac{n!}{r_1!r_2!\cdots(1!)^{r_1}(2!)^{r_2}\cdots}X_1^{r_1}X_2^{r_2}\cdots X_{n-k+1}^{r_{n-k+1}}, \tag{5}$$

the sum to be taken over all elements $(r_1, \ldots, r_{n-k+1})$ of the set $\mathbb{P}(n,k)$ of (n,k)-partition types, i. e., sequences of integers $r_1, r_2, r_3, \ldots \geq 0$ such that $r_1 + r_2 + r_3 + \cdots = k$ and $r_1 + 2r_2 + 3r_3 + \cdots = n$.

Let us now assume that ψ is (compositionally) invertible and go from ψ to its inverse $\overline{\psi}$ in the equation (4). Then it can be shown that there is a doubly indexed family of Laurent polynomials $A_{n,k}$ in the indeterminates $X_1^{-1}, X_2, \ldots, X_{n-k+1}$, so that the analogous identity

$$\frac{1}{k!}\overline{\psi}(x)^k = \sum_{n \geq k} A_{n,k}(\psi_1, \ldots, \psi_{n-k+1}) \frac{x^n}{n!} \tag{6}$$

holds. The polynomial family $A_{n,k}$ also allows a 'diophantine' representation in the manner of (5):

$$A_{n,k} = X_1^{-(2n-1)} \sum_{\mathbb{P}(2n-1-k,n-1)} \frac{(-1)^{n-1-r_1}(2n-2-r_1)!}{(k-1)!r_2!r_3!\cdots(2!)^{r_2}(3!)^{r_3}\cdots} X_1^{r_1} X_2^{r_2} \cdots X_{n-k+1}^{r_{n-k+1}}. \tag{7}$$

Here the sum has to be taken over the set $\mathbb{P}(2n-1-k, n-1)$ of all partitions of $2n-1-k$ elements into $n-1$ non-empty blocks, that is, of all sequences $r_1, r_2, r_3, \ldots$ of non-negative integers such that $r_1 + r_2 + r_3 + \cdots = n-1$ and $r_1 + 2r_2 + 3r_3 + \cdots = 2n-1-k$.

We say that $B_{n,k}$ and $A_{n,k}$ are *inverse* (or *orthogonal*) *companions* of each other because they satisfy $\sum_{j=k}^{n} A_{n,j}B_{j,k} = \delta_{nk}$, where $\delta_{nn} = 1$, $\delta_{nk} = 0$, if $n \neq k$ (Kronecker's symbol). These and other fundamental properties are discussed in detail in [7, 8]; in addition, the reader is referred to the monograph [9]. In it, the collective term *multivariate Stirling polynomials of the first and second kind* was proposed for $A_{n,k}$ and $B_{n,k}$ since the associated coefficient sums $A_{n,k}(1, \ldots, 1)$ and $B_{n,k}(1, \ldots, 1)$ turn out to be the signed Stirling numbers of the first kind and the Stirling numbers of the second kind, respectively.

The multivariate Stirling polynomials have the remarkable property that the $A_{n,k}$ can be expressed in different ways by $B_{n,k}$, and vice versa (which in particular directly implies the well-known corresponding statements for the Stirling numbers of the first and second kind). The relevant formulas can be found in [7, Sect. 6] and as *Generalized Schlömilch-Schläfli identities* in [8, Thm. 5.5].

Here we restrict ourselves to the special case $k = 1$. Equation (6) then states that the nth Taylor coefficient $\overline{\psi}_n$ of the inverse of ψ is represented by $A_{n,1}(\psi_1, \ldots, \psi_n)$. On the other hand, according to [7, Thm. 6.1 ($k = 1$)] we have

$$A_{n,1}(\psi_1, \ldots, \psi_n) = \sum_{r=0}^{n} (-1)^r \psi_1^{-(n+r)} B_{n-1+r,r}(0, \psi_2, \ldots, \psi_n). \tag{8}$$

While Comtet uses the sum on the right-hand side (or a slightly different equivalent) to calculate $\overline{\psi}_n$ (cf., e.g., [1, eq. (7)] and [2, Thm. E, p. 151]), in the present work the explicit expression resulting from (7) takes over this task.

4. The first reduction step

From Section 2 we already know that g is invertible and therefore the inverse can be written as a Taylor series as follows:

$$\overline{g}(y) = \sum_{n \geq 1} \overline{g}_n \frac{y^n}{n!}. \tag{9}$$

Based on this, we obtain the following preliminary result:

Theorem 4.1. *The general coefficient $\overline{g}_n = \overline{g}_n(x)$ in (9) is well-defined as a formal power series with Taylor coefficients $\overline{g}_{n,j}$ ($n \geq 1, j \geq 0$). Using these coefficients, the Taylor coefficients of the implicit* *function $y = y(x)$ in (3) are as follows:*

$$y_m = \sum_{n=1}^{m} \binom{m}{n} \left\{ \sum_{k=1}^{n} (-1)^k \overline{g}_{k,m-n} B_{n,k}(f_{1,0}, \dots, f_{n-k+1,0}) \right\}. \tag{10}$$

Proof. By taking $\psi = g$ and $k = 1$ in (6) we obtain from (2) for every integer $n \geq 1$:

$$\overline{g}_n(x) = A_{n,1}(\varphi_1(x), \dots, \varphi_n(x)). \tag{11}$$

This expression can be expanded into a Taylor series if the right-hand side of (11) is well-defined. A closer look at formula (7) shows that the condition $\varphi_1(x) \neq 0$ (already verified in Section 2) is sufficient for this. We now write $\overline{g}_n$ in series form:

$$\overline{g}_n(x) = \sum_{j \geq 0} \overline{g}_{n,j} \frac{x^j}{j!}.$$

Together with (3) and (9) this leads to the following series for $y(x)$:

$$y(x) = \sum_{j \geq 0} \sum_{k \geq 1} (-1)^k \overline{g}_{k,j} \frac{x^j \varphi_0(x)^k}{j! k!}. \tag{12}$$

To evaluate the factor $\frac{\varphi_0(x)^k}{k!}$, we use (4) and observe $D^j(\varphi_0)(0) = f_{j,0}$. Hence

$$\frac{\varphi_0(x)^k}{k!} = \sum_{n \geq k} B_{n,k}(f_{1,0}, \dots, f_{n-k+1,0}) \frac{x^n}{n!}, \tag{13}$$

and after substituting (13) in (12):

$$\begin{aligned} y(x) &= \sum_{n,j \geq 0} \sum_{k \geq 1} (-1)^k \overline{g}_{k,j} B_{n,k}(f_{1,0}, \dots, f_{n-k+1,0}) \frac{x^{n+j}}{n! j!} \\ &= \sum_{m \geq n \geq 0} \binom{m}{n} \left\{ \sum_{k \geq 1} (-1)^k \overline{g}_{k,m-n} B_{n,k}(f_{1,0}, \dots, f_{n-k+1,0}) \right\} \frac{x^m}{m!}. \end{aligned}$$

Finally, note that the sum in curly brackets is non-zero if and only if $m \geq n \geq k \geq 1$. This proves the asserted equation (10). □

5. Special cases and application example

The result proven in the previous section already shows the outer structure of the coefficient y_m. However, most of the combinatorial complexity is still hidden in the coefficients $\overline{g}_{k,m-n}$. For a calculation *ad hoc*, therefore, only very small values of m can be considered for the time being. So, it makes sense to deal with the cases $m = 1$ and $m = 2$ first, as the resulting statements can then be easily compared with the well-known formulas for the first and second derivative of $y(x)$:

$$y'(x) = -\frac{f_x}{f_y}, \qquad y''(x) = \frac{2 f_x f_{xy}}{f_y^2} - \frac{f_{xx}}{f_y} - \frac{f_x^2 f_{yy}}{f_y^3}. \tag{14}$$

Case $m = 1$: It follows from Theorem 4.1 $y_1 = (-1)^1 \overline{g}_{1,0} B_{1,1}(f_{1,0}) = -\overline{g}_{1,0} f_{1,0}$. Observing $A_{1,1} = X_1^{-1}$ we thus obtain $\overline{g}_{1,0} = \overline{g}_1(0) = A_{1,1}(\varphi_1)(0) = \varphi_1(0)^{-1} = f_{0,1}^{-1}$ and hence $y_1 = -f_{1,0} f_{0,1}^{-1}$ (which results from $y'(x)$ in (14) by taking $x \to 0$).

Case $m = 2$: Even here, the computing effort increases noticeably. We have

$$\begin{aligned} y_2 &= \binom{2}{1} \sum_{k=1}^{1} (-1)^k \overline{g}_{k,1} B_{1,k}(f_{1,0}, \ldots, f_{2-k,0}) \\ &\quad + \binom{2}{2} \sum_{k=1}^{2} (-1)^k \overline{g}_{k,0} B_{2,k}(f_{1,0}, \ldots, f_{3-k,0}) \\ &= -2\overline{g}_{1,1} B_{1,1}(f_{1,0}) - \overline{g}_{1,0} B_{2,1}(f_{1,0}, f_{2,0}) + \overline{g}_{2,0} B_{2,2}(f_{1,0}). \end{aligned}$$

Now recall (from (5) and (7)) $B_{2,1} = X_2$, $B_{2,2} = X_1^2$, $A_{2,1} = -X_1^{-3} X_2$, and observe that $\overline{g}_1'(x) = D(A_{1,1}(\varphi_1))(x) = -\varphi_1'(x)\varphi_1(x)^{-2}$. This yields

$$\begin{aligned} y_2 &= -2\overline{g}_1'(0) f_{1,0} - f_{0,1}^{-1} f_{2,0} + \overline{g}_2(0) f_{1,0}^2 \\ &= 2\frac{\varphi_1'(0)}{\varphi_1(0)^2} f_{1,0} - f_{0,1}^{-1} f_{2,0} - \frac{\varphi_2(0)}{\varphi_1(0)^3} f_{1,0}^2 \\ &= \frac{2 f_{1,1} f_{1,0}}{f_{0,1}^2} - \frac{f_{2,0}}{f_{0,1}} - \frac{f_{0,2} f_{1,0}^2}{f_{0,1}^3}, \end{aligned}$$

which of course also follows from the formula for $y''(x)$ in (14).

Remark. The number of distinct monomials in y_m grows rapidly with m; it is 9 for y_3, 24 for y_4, 61 for y_5, and 91159 for y_{15}. Comtet [2, p. 175] established a generating function for this sequence and gave a table with some of its values. See also Comtet/Fiolet [3] and the correction made by Wilde [11].

Remark. In order to be able to calculate y_m for some $m \geq 3$, it is quite useful to take the instances of the Stirling polynomials from a table or to generate them using a computer algebra program. The reader will find a set of tables for both, the $B_{n,k}$ and $A_{n,k}$, in [7, p. 2471] and [9, p. 51–54]. The partial Bell polynomials $B_{n,k}$ are tabulated in [6, p. 49] and [2, p. 307–308]. The author's *Mathematica* Package for the Multivariate Stirling Polynomials (providing also a rule for replacing indeterminates) can be found at `http://www.gefilde.de/ashome/software/msp/stirling.html`. For more details cf. the appendix in [9, p. 141–145].

Let us now turn to the calculation of some further special coefficients y_m using the formula (10). Equation (11) shows that determining the coefficients $\overline{g}_{n,j}$ needs the higher derivatives (w.r.t. x) of the function that arises from $A_{n,1}$ by replacing its indeterminates $X_1, \ldots, X_n$ with $\varphi_1(x), \ldots, \varphi_n(x)$, respectively. If we now look at (6), it becomes clear that after differentiation (and letting x approach 0), only factors of the form $D^r(\varphi_\nu)(0)$ (and their powers) need to be taken into account. Finally, taking into account the simple fact (readily from the assumptions in Section 2) that

$$D^r(\varphi_\nu)(0) = f_{r,\nu} \quad (r = 0, 1, 2, \ldots),$$

we thus get expressions for $\overline{g}_{n,j}$ exclusively depending on the coefficients $f_{m,n}$.

This method was used here to determine the values of y_3, y_4 (in accordance with Comtet [2, p. 153]) and, in addition to Comtet, also of y_5:

$$\begin{aligned} y_3 =& f_{0,3} f_{1,0}^3 f_{0,1}^{-4} - 3 f_{0,2}^2 f_{1,0}^3 f_{0,1}^{-5} + 9 f_{0,2} f_{1,1} f_{1,0}^2 f_{0,1}^{-4} - 3 f_{1,2} f_{1,0}^2 f_{0,1}^{-3} + 3 f_{2,1} f_{1,0} f_{0,1}^{-2} \\ & - 6 f_{1,1}^2 f_{1,0} f_{0,1}^{-3} - 3 f_{0,2} f_{2,0} f_{1,0} f_{0,1}^{-3} + 3 f_{1,1} f_{2,0} f_{0,1}^{-2} - f_{3,0} f_{0,1}^{-1} \end{aligned}$$

$$\begin{aligned} y_4 =& 10 f_{0,2} f_{0,3} f_{1,0}^4 f_{0,1}^{-6} - f_{0,4} f_{1,0}^4 f_{0,1}^{-5} - 15 f_{0,2}^3 f_{1,0}^4 f_{0,1}^{-7} + 60 f_{0,2}^2 f_{1,1} f_{1,0}^3 f_{0,1}^{-6} + 4 f_{1,3} f_{1,0}^3 f_{0,1}^{-4} \\ & - 16 f_{0,3} f_{1,1} f_{1,0}^3 f_{0,1}^{-5} - 24 f_{0,2} f_{1,2} f_{1,0}^3 f_{0,1}^{-5} + 36 f_{1,1} f_{1,2} f_{1,0}^2 f_{0,1}^{-4} + 6 f_{0,3} f_{2,0} f_{1,0}^2 f_{0,1}^{-4} \\ & + 18 f_{0,2} f_{2,1} f_{1,0}^2 f_{0,1}^{-4} - 6 f_{2,2} f_{1,0}^2 f_{0,1}^{-3} - 72 f_{0,2} f_{1,1}^2 f_{1,0}^2 f_{0,1}^{-5} - 18 f_{0,2}^2 f_{2,0} f_{1,0}^2 f_{0,1}^{-5} \\ & + 24 f_{1,1}^3 f_{1,0} f_{0,1}^{-4} + 36 f_{0,2} f_{1,1} f_{2,0} f_{1,0} f_{0,1}^{-4} + 4 f_{3,1} f_{1,0} f_{0,1}^{-2} - 12 f_{1,2} f_{2,0} f_{1,0} f_{0,1}^{-3} \end{aligned}$$

$$
\begin{aligned}
&- 24f_{1,1}f_{2,1}f_{1,0}f_{0,1}^{-3} - 4f_{0,2}f_{3,0}f_{1,0}f_{0,1}^{-3} + 6f_{2,0}f_{2,1}f_{0,1}^{-2} + 4f_{1,1}f_{3,0}f_{0,1}^{-2} - f_{4,0}f_{0,1}^{-1}\\
&- 3f_{0,2}f_{2,0}^{2}f_{0,1}^{-3} - 12f_{1,1}^{2}f_{2,0}f_{0,1}^{-3}
\end{aligned}
$$

$$
\begin{aligned}
y_5 =&105f_{0,2}^{2}f_{0,3}f_{1,0}^{5}f_{0,1}^{-8} + f_{0,5}f_{1,0}^{5}f_{0,1}^{-6} - 10f_{0,3}^{2}f_{1,0}^{5}f_{0,1}^{-7} - 15f_{0,2}f_{0,4}f_{1,0}^{5}f_{0,1}^{-7} - 105f_{0,2}^{4}f_{1,0}^{5}f_{0,1}^{-9}\\
&+ 525f_{0,2}^{3}f_{1,1}f_{1,0}^{4}f_{0,1}^{-8} + 25f_{0,4}f_{1,1}f_{1,0}^{4}f_{0,1}^{-6} + 50f_{0,3}f_{1,2}f_{1,0}^{4}f_{0,1}^{-6} + 50f_{0,2}f_{1,3}f_{1,0}^{4}f_{0,1}^{-6}\\
&- 5f_{1,4}f_{1,0}^{4}f_{0,1}^{-5} - 300f_{0,2}f_{0,3}f_{1,1}f_{1,0}^{4}f_{0,1}^{-7} - 225f_{0,2}^{2}f_{1,2}f_{1,0}^{4}f_{0,1}^{-7} + 200f_{0,3}f_{1,1}^{2}f_{1,0}^{3}f_{0,1}^{-6}\\
&+ 600f_{0,2}f_{1,1}f_{1,2}f_{1,0}^{3}f_{0,1}^{-6} + 100f_{0,2}f_{0,3}f_{2,0}f_{1,0}^{3}f_{0,1}^{-6} + 150f_{0,2}^{2}f_{2,1}f_{1,0}^{3}f_{0,1}^{-6} + 10f_{2,3}f_{1,0}^{3}f_{0,1}^{-4}\\
&- 60f_{1,2}^{2}f_{1,0}^{3}f_{0,1}^{-5} - 80f_{1,1}f_{1,3}f_{1,0}^{3}f_{0,1}^{-5} - 10f_{0,4}f_{2,0}f_{1,0}^{3}f_{0,1}^{-5} - 40f_{0,3}f_{2,1}f_{1,0}^{3}f_{0,1}^{-5}\\
&- 60f_{0,2}f_{2,2}f_{1,0}^{3}f_{0,1}^{-5} - 900f_{0,2}^{2}f_{1,1}^{2}f_{1,0}^{3}f_{0,1}^{-7} - 150f_{0,2}^{3}f_{2,0}f_{1,0}^{3}f_{0,1}^{-7} + 600f_{0,2}f_{1,1}^{3}f_{1,0}^{2}f_{0,1}^{-6}\\
&+ 450f_{0,2}^{2}f_{1,1}f_{2,0}f_{1,0}^{2}f_{0,1}^{-6} + 30f_{1,3}f_{2,0}f_{1,0}^{2}f_{0,1}^{-4} + 90f_{1,2}f_{2,1}f_{1,0}^{2}f_{0,1}^{-4} + 90f_{1,1}f_{2,2}f_{1,0}^{2}f_{0,1}^{-4}\\
&+ 10f_{0,3}f_{3,0}f_{1,0}^{2}f_{0,1}^{-4} + 30f_{0,2}f_{3,1}f_{1,0}^{2}f_{0,1}^{-4} - 10f_{3,2}f_{1,0}^{2}f_{0,1}^{-3} - 360f_{1,1}^{2}f_{1,2}f_{1,0}^{2}f_{0,1}^{-5}\\
&- 120f_{0,3}f_{1,1}f_{2,0}f_{1,0}^{2}f_{0,1}^{-5} - 180f_{0,2}f_{1,2}f_{2,0}f_{1,0}^{2}f_{0,1}^{-5} - 360f_{0,2}f_{1,1}f_{2,1}f_{1,0}^{2}f_{0,1}^{-5}\\
&- 30f_{0,2}^{2}f_{3,0}f_{1,0}^{2}f_{0,1}^{-5} + 15f_{0,3}f_{2,0}^{2}f_{1,0}f_{0,1}^{-4} + 180f_{1,1}f_{1,2}f_{2,0}f_{1,0}f_{0,1}^{-4} + 180f_{1,1}^{2}f_{2,1}f_{1,0}f_{0,1}^{-4}\\
&+ 90f_{0,2}f_{2,0}f_{2,1}f_{1,0}f_{0,1}^{-4} + 60f_{0,2}f_{1,1}f_{3,0}f_{1,0}f_{0,1}^{-4} + 5f_{4,1}f_{1,0}f_{0,1}^{-2} - 30f_{2,1}^{2}f_{1,0}f_{0,1}^{-3}\\
&- 30f_{2,0}f_{2,2}f_{1,0}f_{0,1}^{-3} - 20f_{1,2}f_{3,0}f_{1,0}f_{0,1}^{-3} - 40f_{1,1}f_{3,1}f_{1,0}f_{0,1}^{-3} - 5f_{0,2}f_{4,0}f_{1,0}f_{0,1}^{-3}\\
&- 120f_{1,1}^{4}f_{1,0}f_{0,1}^{-5} - 45f_{0,2}^{2}f_{2,0}^{2}f_{1,0}f_{0,1}^{-5} - 360f_{0,2}f_{1,1}^{2}f_{2,0}f_{1,0}f_{0,1}^{-5} + 45f_{0,2}f_{1,1}f_{2,0}^{2}f_{0,1}^{-4}\\
&+ 60f_{1,1}^{3}f_{2,0}f_{0,1}^{-4} + 10f_{2,1}f_{3,0}f_{0,1}^{-2} + 10f_{2,0}f_{3,1}f_{0,1}^{-2} + 5f_{1,1}f_{4,0}f_{0,1}^{-2} - f_{5,0}f_{0,1}^{-1}\\
&- 15f_{1,2}f_{2,0}^{2}f_{0,1}^{-3} - 60f_{1,1}f_{2,0}f_{2,1}f_{0,1}^{-3} - 20f_{1,1}^{2}f_{3,0}f_{0,1}^{-3} - 10f_{0,2}f_{2,0}f_{3,0}f_{0,1}^{-3}
\end{aligned}
$$

Example. Finally, we briefly discuss the example of the function

$$f(x,y) := x - y + x^p y^{q+1} \qquad (p, q \text{ integers } \geq 0)$$

presented by Comtet in [2, (IV), p. 153]. It is largely tailored to the application of the Lagrange inversion, but can also be treated just as well with the method described above. We first exclude the trivial case $q = 0$, in which $y = y(x) = x/(1-x^p)$ is immediately apparent as the solution to the equation $f(x,y) = 0$. By assuming $q \geq 1$, we obtain (using the notations from Section 2):

$$\varphi_0(x) = x, \quad \varphi_1(x) = -1, \quad \varphi_{q+1}(x) = (q+1)!x^p, \tag{15}$$

and $\varphi_n(x) = 0$ for every n with $2 \leq n \neq q+1$. This results in the following for the coefficient in (9) and (11):

$$\overline{g}_n(x) = A_{n,1}(\varphi_1(x), 0, \ldots, 0, \varphi_{q+1}(x), 0, \ldots, 0).$$

A closer look at the expression on the right-hand side using (7) reveals that it vanishes except for indices of the form $n = qr + 1$ (with integer $r \geq 0$), in which case we have

$$
\begin{aligned}
\overline{g}_{qr+1}(x) &= (-1)^r \frac{((q+1)r)!}{r!(q+1)!^r} \cdot \frac{\varphi_{q+1}(x)^r}{\varphi_1(x)^{(q+1)r+1}}\\
&= (-1)^{qr+1} x^{pr} \frac{((q+1)r)!}{r!}.
\end{aligned}
$$

From this we read directly that the Taylor coefficients $\overline{g}_{qr+1,l} = D^l(\overline{g}_{qr+1})(0)$ of the function $\overline{g}_{qr+1}(x)$ are just non-zero for $l = pr$, namely

$$\overline{g}_{qr+1,pr} = (-1)^{qr+1}((q+1)r)! \frac{(pr)!}{r!}.$$

To evaluate (10) with these coefficients, we first notice $B_{n,k}(f_{1,0}, f_{2,0}, \ldots, f_{n-k+1,0}) = B_{n,k}(1, 0, \ldots, 0) = \delta_{nk}$ thus obtaining

$$y_m = \sum_{n=1}^{m} \binom{m}{n} (-1)^n \overline{g}_{n,m-n}.$$

Obviously, y_m vanishes except for the values $m = (p+q)r+1$ and $n = qr+1$ ($r \geq 0$ integer). So, after a little calculation the resulting general Taylor coefficient ($\neq 0$) turns out to be just the one reported by Comtet:

$$y_{(p+q)r+1} = \binom{(q+1)r}{r} \frac{((p+q)r+1)!}{qr+1}. \tag{16}$$

Historical Remark. In the special case $p = 0$ and by setting $x = -a \in \mathbb{C}$, $f(x,y) = 0$ converts to an algebraic equation of $(q+1)$th degree: $y^{q+1} - y = a$. As early as 1758, Lambert [5] had provided an infinite series solution to this type of problems. It was later rediscovered by Eisenstein when he was fourteen years old. Stillwell's nice article [10] tells the story of his solution in the case $q = 4$:

$$y = -\sum_{r=0}^{\infty} \binom{5r}{r} \frac{a^{4r+1}}{4r+1}.$$

6. The second reduction step

In the second and final step, we will show how the general Taylor coefficient $\overline{g}_{k,l}$ of $\overline{g}_k(x)$ which appears in Theorem 4.1 can be represented by a polynomial expression depending exclusively on the $f_{m,n}$. As explained in Section 3, we make use of the Stirling polynomials of the first kind to accomplish the series reversion in question. According to (8), this means that one less summation is required.

To facilitate the evaluation of higher-order derivatives of function powers, we first introduce a simple but useful

Auxiliary Statement. *Let h be a function given by the power series $h(x) = \sum_{n \geq 0} h_n \frac{x^n}{n!}$, $h_0 \neq 0$, and let $r, j \in \mathbb{Z}$ with $j \geq 0$. Then the following applies:*

$$D^j(h^r)(0) = \sum_{k=0}^{j} (r)_k h_0^{r-k} B_{j,k}(h_1, \dots, h_{j-k+1}). \tag{17}$$

Proof. We write h^r as a composite function $\mathrm{id}^r \circ h$ and obtain the following using Faà di Bruno's formula [2, Thm. A]:

$$D^j(\mathrm{id}^r \circ h) = \sum_{k=0}^{j} (D^k(\mathrm{id}^r) \circ h) \cdot B_{j,k}(D(h), D^2(h), \dots, D^{j-k+1}(h)).$$

Now, $(D^k(\mathrm{id}^r) \circ h)(0) = 0$ vanishes for $0 \leq r < k$. Otherwise $(D^k(\mathrm{id}^r) \circ h)(0) = (r)_k h_0^{r-k}$, where $(r)_k := r(r-1)\cdots(r-k+1)$ for $k \geq 1$ and $(r)_0 := 1$ denotes the falling factorial. Taking into account $D^i(h)(0) = h_i$ ($i = 1, 2, 3, \dots$), this results in the stated formula (17). □

The expression on the right-hand side of (17) can be written more concisely as $\widehat{P}_{j,r}(h_0, h_1, \dots, h_j)$. Here, $\widehat{P}_{j,r}$ denotes the *potential polynomials* introduced by Comtet [2, p. 141, Eq. (5f)]; see also [8, Eq. (3.10)]. Note that, for negative r, $\widehat{P}_{j,r}$ is a Laurent polynomial in the indeterminates $X_0^{-1}, X_1, \dots, X_j$.

We are now in the position to implement the announced second reduction step.

Theorem 6.1. *Under the assumptions of Theorem 4.1 and for $m \geq n \geq k \geq 1$ we have*

$$\overline{g}_{k,m-n} = \sum_{\mathbb{P}(2k-2,k-1)} \left\{ \frac{(-1)^{k-1-r_1}(2k-2-r_1)!}{r_2! \cdots r_k! (2!)^{r_2} \cdots (k!)^{r_k}} \times \sum_{j_1+\cdots+j_k=m-n} \left\{ \frac{(m-n)!}{j_1! \cdots j_k!} \times \widehat{P}_{j_1,r_1-2k+1}(f_{0,1}, f_{1,1}, \dots, f_{j_1,1}) \prod_{\nu=2}^{k} \widehat{P}_{j_\nu,r_\nu}(f_{0,\nu}, f_{1,\nu}, \dots, f_{j_\nu,\nu}) \right\} \right\}.$$

Proof. Since the derivative D^l of l-th order is a linear operator for every integer $l \geq 0$, we obtain from equation (7):

$$\begin{aligned}\overline{g}_{k,l} &= D^l(\overline{g}_k)(0) = D^l(A_{k,1}(\varphi_1, \ldots, \varphi_k))(0) \\ &= \sum_{\mathbb{P}(2k-2,k-1)} \frac{(-1)^{k-1-r_1}(2k-2-r_1)!}{r_2! \cdots r_k!(2!)^{r_2} \cdots (k!)^{r_k}} D^l(\varphi_1^{r_1-2k+1}\varphi_2^{r_2} \cdots \varphi_k^{r_k})(0).\end{aligned} \tag{18}$$

We evaluate the term $D^l(\ldots)$ by means of the general Leibniz product rule as follows:

$$D^l(\varphi_1^{r_1-2k+1}\varphi_2^{r_2} \cdots \varphi_k^{r_k}) = \sum_{\substack{j_1+j_2+\cdots+j_k=l \\ j_1,j_2,\ldots,j_k \geq 0}} \frac{l!}{j_1!j_2! \cdots j_k!} D^{j_1}(\varphi_1^{r_1-2k+1})D^{j_2}(\varphi_2^{r_2}) \cdots D^{j_k}(\varphi_k^{r_k}). \tag{19}$$

Therefore, only expressions like $D^{j_\nu}(\varphi_\nu^{r_\nu})(0)$ remain to be reduced. We first apply the auxiliary statement (17) to the first factor on the right-hand side of (19):

$$\begin{aligned}D^{j_1}(\varphi_1^{r_1-2k+1})(0) &= \widehat{P}_{j_1,r_1-2k+1}(D^0(\varphi_1)(0), D^1(\varphi_1)(0), \ldots, D^{j_1}(\varphi_1)(0)) \\ &= \widehat{P}_{j_1,r_1-2k+1}(f_{0,1}, f_{1,1}, \ldots, f_{j_1,1}).\end{aligned} \tag{20 a}$$

In a completely analogous manner, we obtain for $\nu \geq 2$:

$$D^{j_\nu}(\varphi_\nu^{r_\nu})(0) = \widehat{P}_{j_\nu,r_\nu}(f_{0,\nu}, f_{1,\nu}, \ldots, f_{j_\nu,\nu}). \tag{20 b}$$

Finally, we obtain the asserted explicit formula for the coefficients $\overline{g}_{k,m-n}$ by putting $l = m - n$ in (18) and (19) and combining this with (20 a,b). ☐

Acknowledgment: I thank the anonymous reviewer for constructive comments and suggestions that significantly improved the manuscript.

References

[1] L. Comtet, Polynômes de Bell et formule explicite des dérivées successives d'une fonction implicite, C. R. Acad. Sci. Paris Série A, t. 267 (1968) 457–460.
[2] L. Comtet: Advanced combinatorics. The art of finite and infinite expansions, rev. and enlarged edition. Reidel, Dordrecht (Holland) 1974.
[3] L. Comtet, M. Fiolet, Sur les dérivées successives d'une fonction implicite, C. R. Acad. Sci. Paris Série A t. 278 (1974) 249–251.
[4] W. P. Johnson, Some problems in differentiation, https://arxiv.org/pdf/1811.12293 version 1 29 Nov 2018.
[5] J. H. Lambert, Observationes variae in mathesin puram. Acta Helvetica 3 (1758) 128–168 (§38 Opera Mathematica 1 16–51, Orell Füssli Verlag, Zürich 1946).
[6] J. Riordan, Introduction to combinatorial analysis, New York, 1958 reprint Dover Publ. Inc. Mineola New York 2002.
[7] A. Schreiber, Multivariate Stirling polynomials of the first and second kind, Discrete Math. 338 (2015) 2462–2484.
[8] A. Schreiber, Inverse relations and reciprocity laws involving partial Bell polynomials and related extensions, Enumer. Combin. Appl. 1(1) (2021) Article S2R3.
[9] A. Schreiber, Stirling polynomials in several indeterminates, Logos Verlag Berlin 2021.
[10] J. Stillwell, Eisenstein's footnote, The Math. Intelligencer 17(2) (1995) 58–62.
[11] T. Wilde, Implicit higher derivatives, and a formula of Comtet and Fiolet. Preprint: https://arXiv.org/pdf/0805.2674, version 1 17 May 2008.
[12] S. Zemel, The Combinatorics of higher derivatives of implicit functions, Monatsh. Math. 188 4 (2019) 765–784. Preprint: https://arxiv.org/abs/1711.07501.