



The modular irregularity strength of middle graphs of path graphs, triangular snake graphs, and diamond graphs

Research Article

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Abstract: Given a simple undirected graph, a modular irregularity labeling is an edge labeling that maps each edge to a positive integer such that the weight of all vertices is distinct. The weight of a vertex is defined as the sum of the label values of the incident edges modulo a certain number, and it is bijective. The minimum value from the range of labelings that allows the graph to have a modular irregularity labeling is known as the graph's modular irregularity strength. This study explicitly establishes the modular irregularity strength for three specific graph classes: the middle graph of a path graph, the triangular snake graph, and the diamond graph, all of which have not been previously studied. The method used in this study involves explicit construction and analysis of the upper and lower bounds of the modular irregularity strength for the graphs. Research on modular irregularity labeling is important theoretically because it can deepen the understanding of more complex graph structures and optimize labeling.

2020 MSC: 05C78 • 05C38 • 05C35

Keywords: Modular irregularity • Middle path graph • Triangular snake graph • Diamond graph

1. Introduction

Graph labeling is one of the most active areas in graph theory, focusing on the assignment of integers to the vertices, edges, or both vertices and edges of a graph according to prescribed rules. Since its

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introduction, graph labeling has developed into a broad research field encompassing numerous labeling schemes, including graceful labeling, harmonious labeling, magic labeling, and irregular labeling (see [12]). Beyond its theoretical significance, graph labeling has found applications in communication networks, coding theory, circuit design, x-ray crystallography, and information security. Among these labeling techniques, irregular labeling has attracted considerable attention because it aims to distinguish graph elements through the assignment of labels that induce distinct weights.

One of the fundamental concepts in graph labeling is irregular labeling, which originated from the study of vertex degrees in graphs. It is well known that every simple graph with at least two vertices contains at least two vertices having the same degree, whereas a multigraph may possess vertices with pairwise distinct degrees. Motivated by this observation, Frieze et al. [11] posed the problem of determining the minimum number of edges that must be added to a simple graph so that the resulting graph becomes irregular. This question inspired the development of irregular graph labeling, where labels are assigned to graph elements in such a way that the induced weights of vertices are distinct. Since then, irregular labeling has evolved into an important branch of graph labeling theory and has generated numerous variants and extensions.

Among the various irregular labeling concepts, irregularity strength is one of the earliest and most extensively studied parameters. In 1988, Chartrand et al. [10] introduced an edge k -labeling $\psi : E(\Gamma) \rightarrow \{1, 2, \dots, k\}$ on a graph $\Gamma = (V(\Gamma), E(\Gamma))$, where k is a positive integer, such that the induced vertex weights are pairwise distinct. For a vertex $u \in V(\Gamma)$, its weight is defined as

$$wt_{\psi}(u) = \sum_{uv \in E(\Gamma)} \psi(uv),$$

namely the sum of the labels assigned to all edges incident with u . The irregularity strength of a graph Γ , denoted by $s(\Gamma)$, is the smallest integer k for which Γ admits an irregular labeling. If no such labeling exists, then $s(\Gamma) = \infty$. Chartrand et al. [10] further showed that $s(\Gamma)$ is finite if and only if Γ contains no isolated edge and has at most one isolated vertex. They also established a general lower bound for the irregularity strength, stated in [Theorem 1.1](#).

Theorem 1.1. [10] Let Γ be a connected graph with $|V(\Gamma)| \geq 3$ that has p_i vertices of degree i , for an integer i . It follows that

$$s(\Gamma) \geq \max_{1 \leq i \leq \Delta(\Gamma)} \left\{ \frac{p_i - 1}{i} + 1 \right\},$$

where $\Delta(\Gamma)$ is the largest degree of vertices in Γ .

Besides the irregularity strength introduced by Chartrand et al. [10], researchers have also investigated total irregular labelings, where both vertices and edges receive labels, leading to the notion of total edge irregularity strength [14] and total vertex irregularity strength [4]. Total edge irregularity strength has also attracted considerable attention in recent years, with exact values being determined for several graph families, including complete graphs and complete bipartite graphs [15], some ladder related graphs [16], quintessence snake graphs [17], uniform theta snake graphs [18], staircase related graphs [24], generalized arithmetic staircase graphs [25], and arithmetic ladder graphs [27].

Other extensions include group irregular labelings, in which labels are taken from algebraic groups and vertex weights are computed using the corresponding group operation [1]. Later, this type of labeling was also studied for disconnected graphs [2]. These developments demonstrate the richness of irregular labeling theory and provide alternative frameworks for distinguishing vertices through induced weights. In particular, modular irregular labeling has recently attracted considerable attention because it combines graph labeling techniques with modular arithmetic and requires the induced vertex weights to form a complete residue system modulo the order of the graph.

Formalizing the concept, Bača et al. [7] introduced modular irregular labeling as follows. Let Γ represent a graph of order n , and let $\mathbb{Z}_n$ symbolize the group of integers under modulo n . An edge k -labeling $\psi : E(\Gamma) \rightarrow \{1, 2, \dots, k\}$ is called as a modular irregular k -labeling if the weight function

$\sigma_{\psi} : V(\Gamma) \rightarrow \mathbb{Z}_n$, defined by $\sigma_{\psi}(u) = \sum_{uv \in E(\Gamma)} \psi(uv)$, for each vertex $u \in V(\Gamma)$, is bijective. The weight $\sigma_{\psi}(u)$ is called as the modular weight of vertex u . The modular irregularity strength of Γ , denoted $ms(\Gamma)$, is the minimum value of k such that Γ admits a modular irregular k -labeling. If no modular irregular labeling exists for the graph Γ , then the modular irregular strength of Γ is defined as $ms(\Gamma) = \infty$.

Bača et al. [7] also established several fundamental properties of modular irregularity strength, including its relationship with ordinary irregularity strength. The following theorem shows that modular irregularity strength imposes a stronger requirement than classical irregularity strength.

Theorem 1.2. [7] If a graph Γ contains no isolated vertices or components isomorphic to K_2 , then $s(\Gamma) \leq ms(\Gamma)$.

Since its introduction, modular irregularity strength has been investigated for numerous graph families. In their pioneering work, Bača et al. [7] determined the exact values of the modular irregularity strength for paths P_n ($n \geq 3$), stars $K_{1,n}$ ($n \geq 2$), cycles C_n ($n \geq 3$), triangular graphs T_n ($n \geq 3$), and gear graphs G_n ($n \geq 3$). Subsequently, they extended these studies to fan graphs F_n [5] and wheel graphs W_n [3]. Furthermore, Barack et al. in [9] studied modular irregularity strength of the corona product of graphs while Sugeng et al. investigated the modular irregularity strength of double-star graphs $S_{k,k}$ and friendship graphs f_n in [20] and some flower graphs in [22]. In [23], Suparta et al. present the modular irregularity strength for some dense graphs including complete graphs and complete bipartite graphs. Recent studies have expanded the scope of modular irregular labeling to more general settings. For example, Bača et al. [6] investigated modular H -irregularity strength, while Barack et al. [8] examined modular irregular labeling for disjoint unions of cycle-related graphs. Very recently, in 2026, Sugeng et al. [21] further contributed to this area by determining the modular irregularity strength for several additional corona products of graphs, complementing and extending previous results. These developments indicate that modular irregular labeling remains an active and evolving area of research.

The study of modular irregularity strength is closely related to the broader development of graph labeling theory, where the objective is to distinguish graph elements through suitable label assignments. Determining exact values of modular irregularity strength is generally difficult, as the existence of a valid labeling depends heavily on the structural properties of the graph. Consequently, each newly investigated graph family contributes to a deeper understanding of the interplay between graph structure and modular weight distributions, while also providing new techniques that may be applicable to other classes of graphs.

Motivated by the continuing search for exact values of modular irregularity strength in new graph families, this paper focuses on three graph classes with distinct structural characteristics, namely the middle graphs of paths, triangular snake graphs, and diamond graphs which definitions are given below.

The middle graph $\mathcal{M}(\Gamma)$ of a connected graph Γ is the graph whose vertex set is defined as $V(\mathcal{M}(\Gamma)) = V(\Gamma) \cup E(\Gamma)$. For any two vertices $x, y \in V(\mathcal{M}(\Gamma))$, vertex x is adjacent to vertex y if and only if either $x, y \in E(\Gamma)$ and x is adjacent to y , or the intersection $\{x, y\} \cap V(\Gamma)$ is non-empty, $\{x, y\} \cap E(\Gamma)$ is non-empty, and the vertex in $\{x, y\} \cap V(\Gamma)$ is incident to the edge in $\{x, y\} \cap E(\Gamma)$ [13].

The triangular snake graph $\mathcal{TS}_n$ is a graph with the vertex set $V(\mathcal{TS}_n) = \{v_1, v_2, \dots, v_n, u_1, \dots, u_{n-1}\}$ and the edges set $E(\mathcal{TS}_n) = \{v_i v_{i+1} : 1 \leq i \leq n\} \cup \{v_i u_i, u_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i v_{i-1}, v_i u_{i-1} : 2 \leq i \leq n\}$ where $n \geq 3$ [26].

Now, let $\mathcal{L}_n$ be the graph isomorphic to $P_n \times P_2$, where $n \geq 2$, representing the ladder graph with the vertex set $V(\mathcal{L}_n) = \{v_i, u_i : 1 \leq i \leq n\}$ and the edge set $E(\mathcal{L}_n) = \{v_i v_{i+1}, u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\}$. By introducing additional edges $u_i v_{i+1}$ for $i = 1, 2, \dots, n-1$ and removing the vertex u_n along with its incident edges $u_{n-1} u_n$ and $u_n v_n$, the resulting graph is referred to as the triangular ladder $\mathcal{TL}_n$. The diamond graph $\mathcal{D}_n$, for $n \geq 3$, is formed by connecting a single vertex x to all vertices v_i , where $i = 1, 2, \dots, n$, of the triangular ladder $\mathcal{TL}_n$ (see [19]). The vertex set and the edge set of $\mathcal{D}_n$ respectively are $V(\mathcal{D}_n) = \{x\} \cup \{u_i : 1 \leq i \leq n-1\} \cup \{v_i : 1 \leq i \leq n\}$ and $E(\mathcal{D}_n) = \{u_i u_{i+1} : 1 \leq i \leq n-2\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n-1\} \cup \{u_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i x : 1 \leq i \leq n\}$. From this construction, the number of vertices in $\mathcal{D}_n$ is $|V(\mathcal{D}_n)| = 2n$, and the number of edges is $|E(\mathcal{D}_n)| = 5n - 5$.

These graphs arise naturally from graph transformations and graph extensions, and their vertex-edge configurations differ substantially from those of previously studied graph families. Consequently,

determining their modular irregularity strength requires the construction of new labeling techniques and weight-distribution schemes.

The main objective of this paper is to determine the exact values of the modular irregularity strength of the middle graphs of paths $\mathcal{M}(P_n)$, triangular snake graphs $\mathcal{TS}_n$, and diamond graphs $\mathcal{D}_n$. The obtained results contribute to the growing body of knowledge on modular irregular labeling and provide further insight into the behavior of modular irregularity strength for graph classes arising from graph transformations and extensions.

2. Modular irregularity strength of middle graphs of path graphs

In this section, we determine the modular irregularity strength of the middle graph of a path. Owing to its construction, the middle graph possesses a richer adjacency structure than the underlying path graph, leading to additional constraints on vertex-weight assignments. The following lemma gives the lower bound for $ms(\mathcal{M}(P_n))$.

Lemma 2.1. *For $n \geq 4$, the middle graph $\mathcal{M}(P_n)$ of the path graph P_n satisfies $ms(\mathcal{M}(P_n)) \geq \lfloor \frac{n-1}{2} \rfloor + 1$.*

Proof. By Theorem 1.1, it follows that,

$$\begin{aligned} s(\mathcal{M}(P_n)) &\geq \max_{1 \leq i \leq \Delta(\mathcal{M}(P_n))} \left\{ \frac{p_i - 1}{i} + 1 \right\} \\ s(\mathcal{M}(P_n)) &\geq \max \left\{ 2, \frac{(n-3)}{2} + 1, \frac{1}{3} + 1, \frac{n-4}{4} + 1 \right\} \\ s(\mathcal{M}(P_n)) &\geq \max \left\{ 2, \frac{n-1}{2}, \frac{4}{3}, \frac{n}{4} \right\} \\ s(\mathcal{M}(P_n)) &\geq \max \left\{ 2, \frac{n-1}{2} \right\}. \end{aligned}$$

It is noted that the irregularity strength of a graph $\mathcal{M}(P_n)$, represented by $s(\mathcal{M}(P_n))$, is a positive integer. Therefore, it follows that

$$s(\mathcal{M}(P_n)) = \begin{cases} \lfloor \frac{n-1}{2} \rfloor + 1 & , \text{ if } n \text{ is even,} \\ \lfloor \frac{n-1}{2} \rfloor & , \text{ if } n \text{ is odd.} \end{cases}$$

If n even, then $s(\mathcal{M}(P_n)) = \lceil \frac{n-1}{2} \rceil = \lfloor \frac{n-1}{2} \rfloor + 1$. Furthermore, it will be shown that if n is odd, then $s(\mathcal{M}(P_n)) \neq \lfloor \frac{n-1}{2} \rfloor$. Assume that for an odd n , the following holds: $s(\mathcal{M}(P_n)) = \lfloor \frac{n-1}{2} \rfloor = \frac{n-1}{2}$. This assumption leads to a contradiction as the maximum vertex weight obtained contradicts the total number of vertices, i.e., $|V(\mathcal{M}(P_n))| = 2n - 1$. Therefore, for odd n , we obtain $s(\mathcal{M}(P_n)) = \lfloor \frac{n-1}{2} \rfloor + 1$. Thus, according to Theorem 2, it follows that:

$$ms(\mathcal{M}(P_n)) \geq s(\mathcal{M}(P_n)) \geq \left\lfloor \frac{n-1}{2} \right\rfloor + 1$$

$$ms(\mathcal{M}(P_n)) \geq \left\lfloor \frac{n-1}{2} \right\rfloor + 1.$$

□

With the lower bound established, we now determine the exact modular irregular strength of $\mathcal{M}(P_n)$ by constructing an appropriate labeling.

Theorem 2.2. For $n \geq 2$, the middle graph $\mathcal{M}(P_n)$ of the path graph P_n satisfies

$$ms(\mathcal{M}(P_n)) = \begin{cases} 2 & , \text{ if } n = 2, \\ 3 & , \text{ if } n = 3, \\ \lfloor \frac{n-1}{2} \rfloor + 1 & , \text{ if } n \geq 4. \end{cases}$$

Proof. Consider the graph $\mathcal{M}(P_n)$ with vertex set $V(\mathcal{M}(P_n)) = \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n-1\}$ and edge set $E(\mathcal{M}(P_n)) = \{e_i v_{i+1}, e_i v_i, e_i e_{i+1} : 1 \leq i \leq n-1\} \cup \{e_i e_{i-1} : 2 \leq i \leq n-1\} \cup \{e_{i-1} v_i : 2 \leq i \leq n\}$. The proof proceeds by considering three distinct cases.

- **Case i:** It is evident that $ms(\mathcal{M}(P_2)) = 2$. Let $n = 3$, consider the function $\psi : E(\mathcal{M}(P_3)) \rightarrow \{1, 2, 3\}$ as follows.

$$\begin{aligned} \psi(e_i v_{i+1}) &= i + 1 & , \text{ if } i = 1, \\ \psi(e_{i-1} v_i) &= i & , \text{ if } i = 2, 3, \\ \psi(e_i v_i) &= i & , \text{ if } i = 1, 2, \\ \psi(e_i e_{i+1}) &= i + 1 & , \text{ if } i = 2, \\ \psi(e_i e_{i-1}) &= i & , \text{ if } i = 2. \end{aligned}$$

One can readily check that ψ is a 3-edge labeling on $\mathcal{M}(P_3)$. Moreover,

$$\{\sigma_\psi(u) : u \in V(\mathcal{M}(P_3))\} = \{0, 1, 2, 3, 4\}.$$

Hence, $ms(\mathcal{M}(P_3)) \leq 3$ and we conclude $ms(\mathcal{M}(P_3)) = 3$.

- **Case ii:** For $n \geq 4$ when n is even. We define an edge labeling ψ as the following.

$$\begin{aligned} \psi(e_i v_{i+1}) &= \begin{cases} \frac{n}{2} - i + 1 & , \text{ if } 1 \leq i \leq \frac{n}{2}, \\ 1 & , \text{ if } \frac{n}{2} < i \leq n - 2, \\ \frac{n}{2} & , \text{ if } i = n - 1, \end{cases} \\ \psi(e_{i-1} v_i) &= \begin{cases} \frac{n}{2} - i + 2 & , \text{ if } 2 \leq i \leq \frac{n+2}{2}, \\ 1 & , \text{ if } \frac{n+2}{2} < i \leq n - 1, \\ \frac{n}{2} & , \text{ if } i = n, \end{cases} \\ \psi(e_i v_i) &= \begin{cases} i & , \text{ if } i = 1, \\ \frac{n}{2} & , \text{ if } 2 \leq i \leq \frac{n}{2}, \text{ or } i = n - 1, \\ n - i - 1 & , \text{ if } \frac{n+2}{2} \leq i \leq n - 2 \text{ when } n \geq 6, \end{cases} \\ \psi(e_i e_{i+1}) &= \frac{n}{2}, \text{ if } i = 1, 2, \dots, n - 1, \\ \psi(e_i e_{i-1}) &= \frac{n}{2}, \text{ if } i = 2, 3, \dots, n - 2. \end{aligned}$$

As

$$\begin{aligned} \max\{\psi(e_i v_{i+1}) : 1 \leq i \leq n - 1\} &= \frac{n}{2} \leq \left\lfloor \frac{n-1}{2} \right\rfloor + 1, \\ \max\{\psi(v_i e_{i-1}) : 2 \leq i \leq n\} &= \frac{n}{2} \leq \left\lfloor \frac{n-1}{2} \right\rfloor + 1, \\ \max\{\psi(e_i e_{i-1}) : 2 \leq i \leq n\} &= \frac{n}{2} \leq \left\lfloor \frac{n-1}{2} \right\rfloor + 1, \\ \max\{\psi(e_i v_i) : 1 \leq i \leq n - 1\} &= \frac{n}{2} \leq \left\lfloor \frac{n-1}{2} \right\rfloor + 1, \end{aligned}$$

$$\max\{\psi(e_i e_{i+1}) : 1 \leq i \leq n-1\} = \frac{n}{2} \leq \left\lfloor \frac{n-1}{2} \right\rfloor + 1,$$

we can see that ψ is a $(\lfloor \frac{n-1}{2} \rfloor + 1)$ -labeling. Now, we examine the weights of vertices v_i .

For $i = 1$,

$$\sigma_{\psi}(v_1) = (\psi(e_1 v_1)) \pmod{2n-1} = 1 \pmod{2n-1}.$$

For $i = 2$,

$$\sigma_{\psi}(v_2) = (\psi(v_2 e_1) + \psi(e_2 v_2)) \pmod{2n-1} = n \pmod{2n-1}.$$

For $3 \leq i \leq \frac{n}{2}, n \geq 6$,

$$\sigma_{\psi}(v_i) = (\psi(e_i v_i) + \psi(v_i e_{i-1})) \pmod{2n-1} = (n-i+2) \pmod{2n-1}.$$

For $\frac{n+2}{2} \leq i \leq n-2, n \geq 6$,

$$\sigma_{\psi}(v_i) = (\psi(e_i v_i) + \psi(v_i e_{i-1})) \pmod{2n-1} = (n-i) \pmod{2n-1}.$$

For $i = n-1$,

$$\sigma_{\psi}(v_{n-1}) = (\psi(v_i e_{i-1}) + \psi(e_i v_i)) \pmod{2n-1} = \left(\frac{n+2}{2}\right) \pmod{2n-1}.$$

For $i = n$,

$$\sigma_{\psi}(v_n) = (\psi(v_n e_{n-1})) \pmod{2n-1} = \left(\frac{n}{2}\right) \pmod{2n-1}.$$

Hence, the set of modular weights of vertices $v_i, i = 1, 2, \dots, n$ is

$$\{\sigma_{\psi}(v_i) : 1 \leq i \leq n\} = \{1, 2, 3, \dots, n\}.$$

Now, we evaluate the weights of vertices e_i .

For $i = 1$,

$$\sigma_{\psi}(e_1) = (\psi(e_1 v_1) + \psi(e_1 v_2) + \psi(e_1 e_2)) \pmod{2n-1} = (n+1) \pmod{2n-1}.$$

For $i = 2$,

$$\begin{aligned} \sigma_{\psi}(e_2) &= (\psi(e_2 v_2) + \psi(e_2 v_3) + \psi(e_2 e_1) + \psi(e_2 e_3)) \pmod{2n-1} = (2n-1) \\ &\pmod{2n-1}. \end{aligned}$$

For $3 \leq i \leq \frac{n}{2}, n \geq 6$,

$$\begin{aligned} \sigma_{\psi}(e_i) &= (\psi(e_i e_{i-1}) + \psi(e_i v_i) + \psi(e_i v_{i+1}) + \psi(e_i e_{i+1})) \pmod{2n-1} \\ &= (2n-i+1) \pmod{2n-1}. \end{aligned}$$

For $\frac{n+2}{2} \leq i \leq n-2$,

$$\begin{aligned} \sigma_{\psi}(e_i) &= (\psi(e_i e_{i-1}) + \psi(e_i v_i) + \psi(e_i v_{i+1}) + \psi(e_i e_{i+1})) \pmod{2n-1} \\ &= (2n-i) \pmod{2n-1}. \end{aligned}$$

For $i = n-1$,

$$\begin{aligned} \sigma_{\psi}(e_{n-1}) &= (\psi(e_{n-1} v_{n-1}) + \psi(e_{n-1} v_n) + \psi(e_{n-1} e_{n-2})) \pmod{2n-1} \\ &= \left(\frac{3n}{2}\right) \pmod{2n-1}. \end{aligned}$$

Hence, the set of modular weights of vertices $e_i, i = 1, 2, \dots, n-1$ is

$$\{\sigma_{\psi}(e_i) : 1 \leq i \leq n-1\} = \{0, n+1, n+2, \dots, 2n-2\}.$$

Therefore,

$$\begin{aligned}\{\sigma_{\psi}(u) : u \in V(\mathcal{M}(P_n))\} &= \{\sigma_{\psi}(v_i) : 1 \leq i \leq n\} \cup \{\sigma_{\psi}(e_i) : 1 \leq i \leq n-1\} \\ &= \{1, 2, 3, \dots, n\} \cup \{0, n+1, n+2, \dots, 2n-2\} \\ &= \{0, 1, 2, \dots, 2n-2\}\end{aligned}$$

and thus ψ is a modular irregular labeling for the graph $\mathcal{M}(P_n)$, implying

$$ms(\mathcal{M}(P_n)) \leq \left\lfloor \frac{2n-1}{2} \right\rfloor + 1. \quad (1)$$

By applying (1) and Lemma 2.1, we conclude that $ms(\mathcal{M}(P_n)) = \left\lfloor \frac{2n-1}{2} \right\rfloor + 1$.

- **Case iii:** For $n \geq 4$ when n is odd.

We construct ψ in the following manner.

$$\begin{aligned}\psi(e_i v_{i+1}) &= \begin{cases} \frac{n-2i+3}{2} & , \text{ if } i = 1, 2, \\ \frac{n-1}{2} & , \text{ if } i = 3, n \geq 7, \\ \frac{n-3}{2} & , \text{ if } 4 \leq i \leq 5, n \geq 9, \\ \frac{n-2i+7}{2} & , \text{ if } 6 \leq i \leq \frac{n-1}{2}, n \geq 13, \\ n-i-1 & , \text{ if } \frac{n+1}{2} \leq i \leq n-2, \\ \frac{n+1}{2} & , \text{ if } i = n-1, \end{cases} \\ \psi(e_{i-1} v_i) &= \begin{cases} \frac{n-2i+5}{2} & , \text{ if } i = 2, 3, \\ \frac{n+1}{2} & , \text{ if } i = n, \\ \frac{n-2i+9}{2} & , \text{ if } 6 \leq i \leq \frac{n+1}{2}, n \geq 11, \\ n-i & , \text{ if } \frac{n+3}{2} \leq i \leq n-1, \\ \frac{n-1}{2} & , \text{ if } i = 4, n = 7, \\ \frac{n-2i+7}{2} & , \text{ if } 4 \leq i \leq 5, n \geq 7, \end{cases} \\ \psi(e_i v_i) &= \begin{cases} i & , \text{ if } i = 1, \\ \frac{n+1}{2} & , \text{ if } i = 2, 3 \text{ or } i = n-1, \\ \frac{n-1}{2} & , \text{ if } i = 4, n = 7, \\ \frac{n-1}{2} & , \text{ if } 4 \leq i \leq 5, n > 7, \\ \frac{n-3}{2} & , \text{ if } 6 \leq i \leq \frac{n+1}{2}, n \geq 11, \\ 1 & , \text{ if } \frac{n+3}{2} \leq i \leq n-2, \end{cases} \\ \psi(e_i e_{i+1}) &= \begin{cases} \frac{n+1}{2} & , \text{ if } i \neq 3, \\ 2 & , \text{ if } i = 3, \end{cases} \\ \psi(e_i e_{i-1}) &= \begin{cases} \frac{n+1}{2} & , \text{ if } i \neq 4, \\ 2 & , \text{ if } i = 4. \end{cases}\end{aligned}$$

As

$$\begin{aligned}\max\{\psi(e_i v_{i+1}) : 1 \leq i \leq n-1\} &= \frac{n+1}{2} = \left\lfloor \frac{n-1}{2} \right\rfloor + 1, \\ \max\{\psi(v_i e_{i-1}) : 2 \leq i \leq n\} &= \frac{n+1}{2} = \left\lfloor \frac{n-1}{2} \right\rfloor + 1, \\ \max\{\psi(e_i v_i) : 1 \leq i \leq n-1\} &= \frac{n+1}{2} = \left\lfloor \frac{n-1}{2} \right\rfloor + 1, \\ \max\{\psi(e_i e_{i+1}) : 1 \leq i \leq n-1\} &= \frac{n+1}{2} = \left\lfloor \frac{n-1}{2} \right\rfloor + 1,\end{aligned}$$

$$\max\{\psi(e_i e_{i-1}) : 2 \leq i \leq n-1\} = \frac{n+1}{2} = \left\lfloor \frac{n-1}{2} \right\rfloor + 1,$$

we conclude that ψ is a $(\lfloor \frac{n-1}{2} \rfloor + 1)$ -labeling. We now evaluate the weights of vertices v_i .

For $i = 1$,

$$\sigma_\psi(v_1) = (\psi(e_1 v_1)) \pmod{2n-1} = 1 \pmod{2n-1}.$$

For $i = 2$,

$$\sigma_\psi(v_2) = (\psi(v_2 e_1) + \psi(e_2 v_2)) \pmod{2n-1} = (n+1) \pmod{2n-1}.$$

For $i = 3$,

$$\sigma_\psi(v_3) = (\psi(e_2 v_3) + \psi(e_3 v_3)) \pmod{2n-1} = n \pmod{2n-1}.$$

For $i = 4, n = 7$,

$$\sigma_\psi(v_i) = (\psi(v_i e_{i-1}) + \psi(e_i v_i)) \pmod{2n-1} = 6 \pmod{13}.$$

For $i = 4, 5, n \geq 9$,

$$\sigma_\psi(v_i) = (\psi(v_i e_{i-1}) + \psi(e_i v_i)) \pmod{2n-1} = (n-i+3) \pmod{2n-1}.$$

For $6 \leq i \leq \frac{n+1}{2}, n \geq 11$,

$$\sigma_\psi(v_i) = (\psi(v_i e_{i-1}) + \psi(e_i v_i)) \pmod{2n-1} = (n-i+3) \pmod{2n-1}.$$

For $\frac{n+3}{2} \leq i \leq n-2, n \geq 7$,

$$\sigma_\psi(v_i) = (\psi(v_i e_{i-1}) + \psi(e_i v_i)) \pmod{2n-1} = (n-i+1) \pmod{2n-1}.$$

For $i = n-1$,

$$\sigma_\psi(v_{n-1}) = (\psi(v_{n-1} e_{n-2}) + \psi(e_{n-1} v_{n-1})) \pmod{2n-1} = \left(\frac{n+3}{2}\right) \pmod{2n-1}.$$

For $i = n$,

$$\sigma_\psi(v_n) = (\psi(e_i v_{i+1})) \pmod{2n-1} = \left(\frac{n+1}{2}\right) \pmod{2n-1}.$$

Hence, the set of modular weights of vertices $v_i, i = 1, 2, \dots, n$ is

$$\{\sigma_\psi(v_i) : 1 \leq i \leq n\} = \{1, 3, 4, \dots, n+1\}.$$

The weights of vertices e_i are given as follows.

For $i = 1$,

$$\sigma_\psi(e_1) = (\psi(e_1 v_1) + \psi(e_1 v_2) + \psi(e_1 e_2)) \pmod{2n-1} = (n+2) \pmod{2n-1}.$$

For $i = 2$,

$$\sigma_\psi(e_2) = (\psi(e_2 v_2) + \psi(e_2 v_3) + \psi(e_2 e_1) + \psi(e_2 e_3)) \pmod{2n-1} = (2n+1) \pmod{2n-1}.$$

For $i = 3, n = 5$,

$$\sigma_\psi(e_3) = (\psi(e_3 e_2) + \psi(e_3 v_3) + \psi(e_3 v_4) + \psi(e_3 e_4)) \pmod{2n-1} = 9 \pmod{9}.$$

For $i = 3, n \geq 7$,

$$\sigma_\psi(e_3) = (\psi(e_3 e_2) + \psi(e_3 v_3) + \psi(e_3 v_4) + \psi(e_3 e_4)) \pmod{2n-1} = \left(\frac{3n+5}{2}\right) \pmod{2n-1}.$$

For $i = 4, n \geq 7$,

$$\sigma_{\psi}(e_4) = (\psi(e_4e_3) + \psi(e_4v_4) + \psi(e_4v_5) + \psi(e_4e_5)) \pmod{2n-1} = \left(\frac{3n+1}{2}\right) \pmod{2n-1}.$$

For $i = 5, n = 7$,

$$\sigma_{\psi}(e_5) = (\psi(e_5e_4) + \psi(e_5v_5) + \psi(e_5v_6) + \psi(e_5e_6)) \pmod{2n-1} = 10 \pmod{13}.$$

For $i = 5, n \geq 9$,

$$\sigma_{\psi}(e_5) = (\psi(e_5e_4) + \psi(e_5v_5) + \psi(e_5v_6) + \psi(e_5e_6)) \pmod{2n-1} = (2n-1) \pmod{2n-1}.$$

For $6 \leq i \leq \frac{n-1}{2}, n \geq 13$,

$$\sigma_{\psi}(e_i) = (\psi(e_ie_{i-1}) + \psi(e_iv_i) + \psi(e_iv_{i+1}) + \psi(e_ie_{i+1})) \pmod{2n-1} = (2n-i+3) \pmod{2n-1}.$$

For $i = \frac{n+1}{2}, n \geq 11$,

$$\sigma_{\psi}(e_i) = (\psi(e_ie_{i-1}) + \psi(e_iv_i) + \psi(e_iv_{i+1}) + \psi(e_ie_{i+1})) \pmod{2n-1} = (2n-2) \pmod{2n-1}.$$

For $\frac{n+3}{2} \leq i \leq n-2, n \geq 9$,

$$\sigma_{\psi}(e_i) = (\psi(e_ie_{i-1}) + \psi(e_iv_i) + \psi(e_iv_{i+1}) + \psi(e_ie_{i+1})) \pmod{2n-1} = (2n-i+1) \pmod{2n-1}.$$

For $i = n-1, n = 5$,

$$\sigma_{\psi}(e_{n-1}) = (\psi(e_ie_{i-1}) + \psi(e_iv_i) + \psi(e_iv_{i+1})) \pmod{2n-1} = 8 \pmod{9}.$$

For $i = n-1, n \geq 7$,

$$\sigma_{\psi}(e_{n-1}) = (\psi(e_ie_{i-1}) + \psi(e_iv_i) + \psi(e_iv_{i+1})) \pmod{2n-1} = \left(\frac{3n+3}{2}\right) \pmod{2n-1}.$$

Hence, the set of modular weights of vertices $e_i, i = 1, 2, \dots, n-1$ is

$$\{\sigma_{\psi}(e_i) : 1 \leq i \leq n-1\} = \{0, 2, n+2, n+3, \dots, 2n-2\}.$$

We obtain

$$\begin{aligned} \{\sigma_{\psi}(u) : u \in V(\mathcal{M}(P_n))\} &= \{\sigma_{\psi}(v_i) : 1 \leq i \leq n\} \cup \{\sigma_{\psi}(e_i) : 1 \leq i \leq n-1\} \\ &= \{1, 3, 4, \dots, n+1\} \cup \{0, 2, n+2, n+3, \dots, 2n-2\} \\ &= \{0, 1, 2, \dots, 2n-2\}. \end{aligned}$$

Therefore, ψ fulfills the requirement for a modular irregular labeling of $\mathcal{M}(P_n)$, and thus

$$ms(\mathcal{M}(P_n)) \leq \left\lfloor \frac{2n-1}{2} \right\rfloor + 1. \quad (2)$$

By applying (2) and Lemma 2.1, we have $ms(\mathcal{M}(P_n)) = \left\lfloor \frac{2n-1}{2} \right\rfloor + 1$.

□

Fig. 1 illustrates a specific instance of the modular irregular labeling for the middle graph of a path, $\mathcal{M}(P_n)$, as described in Theorem 2.2.

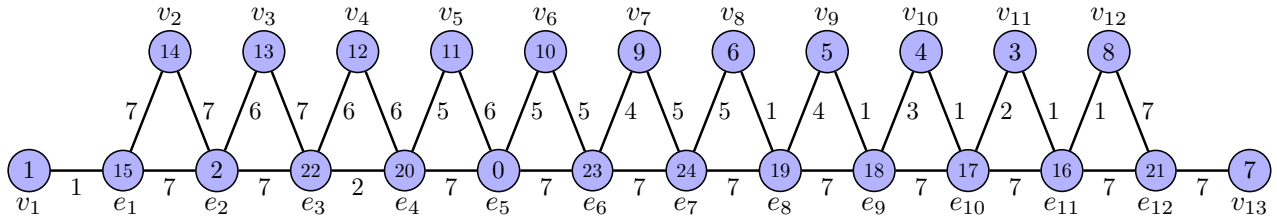


Figure 1. Irregular Labeling of Graphs $\mathcal{M}(P_{13})$

3. Modular irregularity strength of triangular snake graphs

Having determined the modular irregularity strength of the middle graph of a path, we now turn our attention to triangular snake graphs. The graph $\mathcal{TS}_n$ exhibits a significantly different structure, consisting of a sequence of adjacent triangles. In this section, we establish the exact value of $ms(\mathcal{TS}_n)$ by deriving an appropriate lower bound and constructing a modular irregular labeling that achieves the bound.

Lemma 3.1. *Let $\mathcal{TS}_n$ be a triangular snake graph with $n \geq 3$. It follows that $ms(\mathcal{TS}_n) \geq \lfloor \frac{n+2}{2} \rfloor + 1$.*

Proof. By Theorem 1.1, it follows that,

$$\begin{aligned} s(\mathcal{TS}_n) &\geq \max_{1 \leq i \leq \Delta(\mathcal{TS}_n)} \left\{ \frac{p_i - 1}{i} + 1 \right\} \\ s(\mathcal{TS}_n) &\geq \max \left\{ \frac{n}{2} + 1, \frac{n-3}{4} + 1 \right\} \\ s(\mathcal{TS}_n) &\geq \max \left\{ \frac{n+2}{2}, \frac{n+1}{4} \right\} \\ s(\mathcal{TS}_n) &\geq \frac{n+2}{2}, \end{aligned}$$

implying

$$s(\mathcal{TS}_n) = \begin{cases} \lfloor \frac{n+2}{2} \rfloor + 1 & , \text{ if } n \text{ is odd,} \\ \lfloor \frac{n+2}{2} \rfloor & , \text{ if } n \text{ is even.} \end{cases}$$

For odd values of n , $s(\mathcal{TS}_n) = \lceil \frac{n+2}{2} \rceil = \lfloor \frac{n+2}{2} \rfloor + 1$. Furthermore, it will be shown that for even n , assuming $s(\mathcal{TS}_n) = \frac{n+2}{2}$ leads to a contradiction because the maximum node weight obtained conflicts with the total number of nodes, $|V(\mathcal{TS}_n)| = 2n-1$. Therefore, for even n , $s(\mathcal{TS}_n) = \lfloor \frac{n+2}{2} \rfloor + 1$. Therefore, according to Theorem 2, we have:

$$ms(\mathcal{TS}_n) \geq s(\mathcal{TS}_n) \geq \left\lfloor \frac{n+2}{2} \right\rfloor + 1.$$

$$ms(\mathcal{TS}_n) \geq \left\lfloor \frac{n+2}{2} \right\rfloor + 1.$$

□

The following theorem provides the precise value of the modular irregularity strength of triangular snake graphs.

Theorem 3.2. Let $\mathcal{TS}_n$ be a triangular snake graph of order $n \geq 3$. Then

$$ms(\mathcal{TS}_n) = \left\lfloor \frac{n+2}{2} \right\rfloor + 1.$$

Proof. Consider the graph $\mathcal{TS}_n$ with vertex set $V(\mathcal{TS}_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n-1\}$ and edge set $E(\mathcal{TS}_n) = \{v_i v_{i+1} : 1 \leq i \leq n\} \cup \{v_i u_i, u_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i v_{i-1}, v_i u_{i-1} : 2 \leq i \leq n\}$. The proof proceeds by considering two distinct cases.

1. **Case i:** Assume that n is even. For $n \geq 3$, the edge labeling ψ is defined as follows.

$$\begin{aligned} \psi(v_i v_{i+1}) &= \begin{cases} 1 & , \text{ if } i = 1, 2, \\ \frac{n}{2} - i + 2 \lfloor \frac{i}{2} \rfloor & , \text{ if } 3 \leq i \leq n-2, n \geq 6, \\ \frac{n+2}{2} & , \text{ if } i = n-1, \end{cases} \\ \psi(v_i v_{i-1}) &= \begin{cases} 1 & , \text{ if } i = 2, 3, \\ (\frac{n}{2} - 1) + \lfloor \frac{i+1}{2} \rfloor - \lfloor \frac{i}{2} \rfloor & , \text{ if } 4 \leq i \leq n-1, n \geq 6, \\ \frac{n+2}{2} & , \text{ if } i = n, \end{cases} \\ \psi(v_i u_i) &= \begin{cases} i & , \text{ if } i = 1, 2, \\ \frac{n-2i+10}{2} & , \text{ if } 3 \leq i \leq \frac{n-2}{2}, n \geq 8, \\ \frac{n+4}{2} & , \text{ if } \frac{n}{2} \leq i \leq n-2, n \geq 6 \text{ or } i = n-1, \end{cases} \\ \psi(u_i v_{i+1}) &= \begin{cases} i+1 & , \text{ if } i = 1, 2, \\ 1 & , \text{ if } 3 \leq i \leq \frac{n}{2} - 1, n \geq 8, \\ i - \frac{n}{2} + 2 & , \text{ if } \frac{n}{2} \leq i \leq n-2, n \geq 6, \\ \frac{n+4}{2} & , \text{ if } i = n-1, \end{cases} \\ \psi(v_i u_{i-1}) &= \begin{cases} i & , \text{ if } i = 2, 3, \\ 1 & , \text{ if } 4 \leq i \leq \frac{n}{2}, n \geq 8, \\ i - \frac{n}{2} + 1 & , \text{ if } \frac{n+2}{2} \leq i \leq n-1, n \geq 6 \\ \frac{n}{2} + 2 & , \text{ if } i = n. \end{cases} \end{aligned}$$

As

$$\begin{aligned} \max\{\psi(v_i v_{i+1}) : 1 \leq i \leq n-1\} &= \frac{n+2}{2} \leq \left\lfloor \frac{n+2}{2} \right\rfloor + 1, \\ \max\{\psi(v_i v_{i-1}) : 2 \leq i \leq n\} &= \frac{n+2}{2} \leq \left\lfloor \frac{n+2}{2} \right\rfloor + 1, \\ \max\{\psi(v_i u_i) : 1 \leq i \leq n-1\} &= \frac{n+4}{2} = \left\lfloor \frac{n+2}{2} \right\rfloor + 1, \\ \max\{\psi(v_i u_{i-1}) : 1 \leq i \leq n-1\} &= \frac{n+4}{2} = \left\lfloor \frac{n+2}{2} \right\rfloor + 1, \\ \max\{\psi(u_i v_{i+1}) : 1 \leq i \leq n-1\} &= \frac{n+4}{2} = \left\lfloor \frac{n+2}{2} \right\rfloor + 1, \end{aligned}$$

ψ is an $(\lfloor \frac{n+2}{2} \rfloor + 1)$ -labeling. Next, we examine the vertex weights. First, we evaluate the weights of vertices v_i .

For $i = 1$,

$$\sigma_\psi(v_1) = (\psi(v_1 v_2) + \psi(v_1 u_1)) \pmod{2n-1} = 2 \pmod{2n-1}.$$

For $i = 2$,

$$\sigma_{\psi}(v_2) = (\psi(v_2v_1) + \psi(v_2v_3) + \psi(v_2u_1) + \psi(v_2u_2)) \pmod{2n-1} = 6 \pmod{2n-1}.$$

For $i = 3, n = 4$,

$$\sigma_{\psi}(v_3) = (\psi(v_3v_2) + \psi(v_3v_4) + \psi(v_3u_2) + \psi(v_3u_3)) \pmod{2n-1} = 11 \pmod{7}.$$

For $i = 3, n = 6$,

$$\sigma_{\psi}(v_3) = (\psi(v_3v_2) + \psi(v_3v_4) + \psi(v_3u_2) + \psi(v_3u_3)) \pmod{2n-1} = 11 \pmod{11}.$$

For $i = 3, n \geq 8$,

$$\sigma_{\psi}(v_3) = (\psi(v_3v_2) + \psi(v_3v_4) + \psi(v_3u_2) + \psi(v_3u_3)) \pmod{2n-1} = (n+5) \pmod{2n-1}.$$

For $i = 4, n = 8$

$$\sigma_{\psi}(v_4) = (\psi(v_4v_3) + \psi(v_4v_5) + \psi(v_4u_3) + \psi(v_4u_4)) \pmod{2n-1} = 14 \pmod{15}.$$

For $4 \leq i \leq \frac{n-2}{2}, n \geq 10$,

$$\begin{aligned} \sigma_{\psi}(v_i) &= (\psi(v_iv_{i-1}) + \psi(v_iv_{i+1}) + \psi(v_iu_{i-1}) + \psi(v_iu_i)) \pmod{2n-1} \\ &= \left(\frac{3n}{2} + 5 - i \right) \pmod{2n-1}. \end{aligned}$$

For $i = \frac{n}{2}, n \geq 10$,

$$\begin{aligned} \sigma_{\psi}(v_i) &= (\psi(v_iv_{i-1}) + \psi(v_iv_{i+1}) + \psi(v_iu_{i-1}) + \psi(v_iu_i)) \pmod{2n-1} \\ &= \left(\frac{3n}{2} + 2 \right) \pmod{2n-1}. \end{aligned}$$

For $\frac{n+2}{2} \leq i \leq n-4, n \geq 10$,

$$\begin{aligned} \sigma_{\psi}(v_i) &= (\psi(v_iv_{i-1}) + \psi(v_iv_{i+1}) + \psi(v_iu_{i-1}) + \psi(v_iu_i)) \pmod{2n-1} \\ &= (n+2-i) \pmod{2n-1}. \end{aligned}$$

For $i = n-3, n \geq 8$,

$$\begin{aligned} \sigma_{\psi}(v_{n-3}) &= (\psi(v_iv_{i-1}) + \psi(v_iv_{i+1}) + \psi(v_iu_{i-1}) + \psi(v_iu_i)) \pmod{2n-1} \\ &= (2n-1) \pmod{2n-1}. \end{aligned}$$

For $i = n-2, n \geq 6$,

$$\begin{aligned} \sigma_{\psi}(v_{n-2}) &= (\psi(v_iv_{i-1}) + \psi(v_iv_{i+1}) + \psi(v_iu_{i-1}) + \psi(v_iu_i)) \pmod{2n-1} \\ &= 2n \pmod{2n-1}. \end{aligned}$$

For $i = n-1, n \geq 6$,

$$\begin{aligned} \sigma_{\psi}(v_{n-1}) &= (\psi(v_iv_{i-1}) + \psi(v_iv_{i+1}) + \psi(v_iu_{i-1}) + \psi(v_iu_i)) \pmod{2n-1} \\ &= 2n+3 \pmod{2n-1}. \end{aligned}$$

For $i = n$,

$$\sigma_{\psi}(v_n) = (\psi(v_nv_{n-1}) + \psi(v_nu_{n-1})) \pmod{2n-1} = (n+3) \pmod{2n-1}.$$

Hence, the set of modular weights of vertices $v_i, i = 1, 2, \dots, n$ is

$$\{\sigma_{\psi}(v_i) : 1 \leq i \leq n\} = \{0, 1, 2, 4, 6, \dots, n+4\}.$$

Then, we evaluate the weights of vertices u_i .

For $i = 1, 2$,

$$\sigma_{\psi}(u_i) = (\psi(u_iv_i) + \psi(u_iv_{i+1})) \pmod{2n-1} = (2i+1) \pmod{2n-1}.$$

$$\text{For } 3 \leq i \leq \frac{n-2}{2}, n \geq 8,$$

$$\sigma_{\psi}(u_i) = (\psi(u_i v_i) + \psi(u_i v_{i+1})) \pmod{2n-1} = \left(\frac{n-2i+12}{2} \right) \pmod{2n-1}.$$

$$\text{For } \frac{n}{2} \leq i \leq n-2, n \geq 6,$$

$$\sigma_{\psi}(u_i) = (\psi(u_i v_i) + \psi(u_i v_{i+1})) \pmod{2n-1} = (i+4) \pmod{2n-1}.$$

$$\text{For } i = n-1,$$

$$\sigma_{\psi}(u_i) = (\psi(u_i v_i) + \psi(u_i v_{i+1})) \pmod{2n-1} = (n+4) \pmod{2n-1}.$$

Hence, the set of modular weights of vertices $u_i, i = 1, 2, \dots, n-1$ is

$$\{\sigma_{\psi}(u_i) : 1 \leq i \leq n-1\} = \{3, 5, 7, 8, \dots, 2n-2\}.$$

So,

$$\begin{aligned} \{\sigma_{\psi}(e) : e \in V(\mathcal{TS}_n)\} &= \{\sigma_{\psi}(v_i) : 1 \leq i \leq n\} \cup \{\sigma_{\psi}(u_i) : 1 \leq i \leq n-1\} \\ &= \{0, 1, 2, 4, \dots, n+4\} \cup \{3, 5, 7, 8, \dots, 2n-2\} \\ &= \{0, 1, 2, \dots, 2n-2\}. \end{aligned}$$

Each vertex is assigned a unique weight from the consecutive integers in $\{2, 3, 4, \dots, 2n\}$. Given that $2n-1$ is congruent to 0 modulo $2n-1$ and $2n$ congruent to 1 modulo $2n-1$, the resulting modular weights are $\{0, 1, 2, \dots, 2n-2\}$. Consequently, the labeling ψ satisfies the criteria for a modular irregular $\lfloor \frac{n+2}{2} + 1 \rfloor$ -labeling.

2. **Case ii:** Suppose that n is odd. Define an edge labeling ψ such that

$$\begin{aligned} \psi(v_i u_i) &= \begin{cases} 1 & , \text{ if } i = 1, 2, \\ 4 & , \text{ if } i = 2, 3, i = n-1, \\ \frac{n+1}{2} - (i-3) & , \text{ if } 3 \leq i \leq \frac{n-3}{2}, n \geq 9, \\ \frac{n+3}{2} & , \text{ if } \frac{n-1}{2} \leq i \leq n-1, n \geq 7, \end{cases} \\ \psi(v_i v_{i+1}) &= \begin{cases} i & , \text{ if } i = 1, 2, n = 3, \\ 1 & , \text{ if } i = 1, 2, n \geq 5, \\ i-2 & , \text{ if } i = 3, 4, n = 5, \\ \left(\frac{2n+i-11}{2} \right) \left(2 \left(\frac{i}{2} - \left\lfloor \frac{i}{2} \right\rfloor \right) \right) & , \text{ if } 4 \leq i \leq \frac{n+3}{2} + 1, n = 7, 9, \\ \quad + \left(1 - 2 \left(\frac{i}{2} - \left\lfloor \frac{i}{2} \right\rfloor \right) \right) & \\ \frac{n-5}{2} + 4 \left\lfloor \frac{i+1}{2} \right\rfloor - 4 \left\lfloor \frac{i}{2} \right\rfloor & , \text{ if } 3 \leq i \leq \frac{n-7}{2}, n \geq 15, \\ (3 + 2 \lceil \frac{n-13}{4} \rceil) (1 - \lceil \frac{i}{2} - \lfloor \frac{i}{2} \rfloor \rceil) & , \text{ if } 5 + 2 \lfloor \frac{n-15}{4} \rfloor \leq i \leq 8 + 2 \lceil \frac{n-13}{4} \rceil, \\ \quad + (8 + 2 \lfloor \frac{n-13}{4} \rfloor) (\lceil \frac{i}{2} - \lfloor \frac{i}{2} \rfloor \rceil) & , n \geq 11, \\ (4 + \lfloor \frac{n-13}{2} \rfloor) (1 - \lceil \frac{i}{2} - \lfloor \frac{i}{2} \rfloor \rceil) & , \text{ if } \frac{n+5}{2} \leq i \leq n-3, n \geq 13, \\ \quad + (8 + \lfloor \frac{n-13}{2} \rfloor) (\lceil \frac{i}{2} - \lfloor \frac{i}{2} \rfloor \rceil) & \\ \frac{n+3}{2} & , \text{ if } i = n-2, n \geq 9, \\ 1 & , \text{ if } i = n-1, n \geq 7, \end{cases} \end{aligned}$$

$$\begin{aligned}\psi(v_i v_{i-1}) &= \begin{cases} i-1 & , \text{ if } i=2, 3, n=3, \\ 1 & , \text{ if } i=2, 3, n \geq 5, \\ i-3 & , \text{ if } i=4, 5, n=5, \\ \left(\frac{2n+i-12}{2}\right) \left(1-2\left(\frac{i}{2}-\lfloor \frac{i}{2} \rfloor\right)\right) & , \text{ if } 4 \leq i \leq \frac{n+3}{2}+1, n=7, 9, \\ \quad + \left(2\left(\frac{i}{2}-\lfloor \frac{i}{2} \rfloor\right)\right) & \\ \frac{n+3}{2} + 4\lfloor \frac{i}{2} \rfloor - 4\lfloor \frac{i+1}{2} \rfloor & , \text{ if } 4 \leq i \leq 5+2\lfloor \frac{n-15}{4} \rfloor, n \geq 15, \\ (8+2\lfloor \frac{n-13}{4} \rfloor)(1-\lceil \frac{i}{2}-\lfloor \frac{i}{2} \rfloor \rceil) & , \text{ if } 6+2\lfloor \frac{n-15}{4} \rfloor \leq i \leq 9+2\lceil \frac{n-13}{4} \rceil, \\ \quad + (3+2\lceil \frac{n-13}{4} \rceil)(\lceil \frac{i}{2}-\lfloor \frac{i}{2} \rfloor \rceil) & n \geq 11, \\ (8+\lfloor \frac{n-13}{2} \rfloor)(1-\lceil \frac{i}{2}-\lfloor \frac{i}{2} \rfloor \rceil) & , \text{ if } \frac{n+7}{2} \leq i \leq n-2, n \geq 13, \\ \quad + (4+\lfloor \frac{n-13}{2} \rfloor)(\lceil \frac{i}{2}-\lfloor \frac{i}{2} \rfloor \rceil) & \\ \frac{n+3}{2} & , \text{ if } i=n-1, n \geq 9, \\ 1 & , \text{ if } i=n, n \geq 7, \end{cases} \\ \psi(u_i v_{i+1}) &= \begin{cases} i+1 & , \text{ if } i=1, 2, \\ i & , \text{ if } i=3, 4, n=5, \\ 1 & , \text{ if } 3 \leq i \leq \frac{n-3}{2}, n \geq 9, \\ \frac{2i-n+5}{2} & , \text{ if } \frac{n-1}{2} \leq i \leq n-1, n \geq 7, \end{cases} \\ \psi(v_i u_{i-1}) &= \begin{cases} i & , \text{ if } i=2, 3, \\ i-1 & , \text{ if } i=4, 5, n=5, \\ 1 & , \text{ if } 4 \leq i \leq \frac{n-1}{2}, n \geq 9, \\ \frac{2i-n+3}{2} & , \text{ if } \frac{n+1}{2} \leq i \leq n, n \geq 7. \end{cases}\end{aligned}$$

As,

$$\begin{aligned}\max\{\psi(v_i v_{i+1}) : 1 \leq i \leq n-1\} &= \frac{n+3}{2} \leq \left\lfloor \frac{n+2}{2} \right\rfloor + 1, \\ \max\{\psi(v_i v_{i-1}) : 2 \leq i \leq n\} &= \frac{n+3}{2} \leq \left\lfloor \frac{n+2}{2} \right\rfloor + 1, \\ \max\{\psi(v_i u_i) : 1 \leq i \leq n-1\} &= \frac{n+3}{2} \leq \left\lfloor \frac{n+2}{2} \right\rfloor + 1, \\ \max\{\psi(v_i u_{i-1}) : 1 \leq i \leq n-1\} &= \frac{n+1}{2} + 1 \leq \left\lfloor \frac{n+2}{2} \right\rfloor + 1, \\ \max\{\psi(u_i v_{i+1}) : 1 \leq i \leq n-1\} &= \frac{n+1}{2} + 1 \leq \left\lfloor \frac{n+2}{2} \right\rfloor + 1,\end{aligned}$$

ψ is an $\lfloor \frac{n+2}{2} \rfloor + 1$ -labeling. Now, we evaluate the weights of vertices v_i .

$$\begin{aligned}\text{For } i=1, \\ \sigma_\psi(v_1) &= (\psi(v_i v_{i+1}) + \psi(v_i u_i)) \pmod{2n-1} = 2 \pmod{2n-1}. \\ \text{For } i=2, n \geq 3, \\ \sigma_\psi(v_2) &= (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\ &= 6 \pmod{5}. \\ \text{For } i=2, n \geq 5, \\ \sigma_\psi(v_2) &= (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\ &= 5 \pmod{2n-1}.\end{aligned}$$

$$\begin{aligned}
 &\text{For } i = 3, n = 3, \\
 &\sigma_{\psi}(v_3) = (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\
 &\quad = 5 \pmod{5}. \\
 &\text{For } i = 3, n = 5, \\
 &\sigma_{\psi}(v_3) = (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\
 &\quad = 9 \pmod{9}. \\
 &\text{For } i = 3, n = 7, \\
 &\sigma_{\psi}(v_3) = (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\
 &\quad = 12 \pmod{13}. \\
 &\text{For } i = 3, n = 9, \\
 &\sigma_{\psi}(v_3) = (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\
 &\quad = 14 \pmod{17}. \\
 &\text{For } i = 4, n = 5, \\
 &\sigma_{\psi}(v_4) = (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\
 &\quad = 10 \pmod{9}. \\
 &\text{For } i = 4, n = 7, \\
 &\sigma_{\psi}(v_4) = (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\
 &\quad = 11 \pmod{13}. \\
 &\text{For } i = 4, n = 9, \\
 &\sigma_{\psi}(v_4) = (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\
 &\quad = 13 \pmod{17}. \\
 &\text{For } i = 4, n = 11, \\
 &\sigma_{\psi}(v_4) = (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\
 &\quad = 15 \pmod{21}. \\
 &\text{For } i = 5, n = 5, \\
 &\sigma_{\psi}(v_5) = (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\
 &\quad = 6 \pmod{9}. \\
 &\text{For } i = 5, n = 7, \\
 &\sigma_{\psi}(v_5) = (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\
 &\quad = 13 \pmod{13}. \\
 &\text{For } i = 5, n = 9, \\
 &\sigma_{\psi}(v_5) = (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\
 &\quad = 15 \pmod{17}. \\
 &\text{For } i = 6, n = 9, \\
 &\sigma_{\psi}(v_6) = (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\
 &\quad = 16 \pmod{17}. \\
 &\text{For } i = 7, n = 9, \\
 &\sigma_{\psi}(v_7) = (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\
 &\quad = 17 \pmod{17}.
 \end{aligned}$$

For $i = 3, n \geq 11$,

$$\begin{aligned}\sigma_{\psi}(v_3) &= (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\ &= (n+6) \pmod{2n-1}.\end{aligned}$$

For $\frac{n-5}{2} \leq i \leq \frac{n-3}{2}, n \geq 13$,

$$\begin{aligned}\sigma_{\psi}(v_i) &= (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\ &= \left(\frac{3n-2i+5}{2}\right) \pmod{2n-1}.\end{aligned}$$

For $4 \leq i \leq \frac{n-7}{2}, n \geq 15$,

$$\begin{aligned}\sigma_{\psi}(v_i) &= (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\ &= \left(\frac{3n-2i+7}{2}\right) \pmod{2n-1}.\end{aligned}$$

For $i = \frac{n-1}{2}, n \geq 11$,

$$\begin{aligned}\sigma_{\psi}(v_i) &= (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\ &= \left(\frac{3n+1}{2}\right) \pmod{2n-1}.\end{aligned}$$

For $\frac{n+1}{2} \leq i \leq \frac{n+5}{3}, n \geq 11$,

$$\begin{aligned}\sigma_{\psi}(v_i) &= (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\ &= 22 \pmod{21}.\end{aligned}$$

For $\frac{n+7}{2} \leq i \leq n-2, n \geq 13$,

$$\begin{aligned}\sigma_{\psi}(v_i) &= (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\ &= \left(15 + 2 \left\lfloor \frac{n-13}{2} \right\rfloor + i\right) \pmod{2n-1}.\end{aligned}$$

For $i = 6, n = 7$,

$$\begin{aligned}\sigma_{\psi}(v_6) &= (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\ &= 14 \pmod{13}.\end{aligned}$$

For $i = n-1, n \geq 9$,

$$\begin{aligned}\sigma_{\psi}(v_{n-1}) &= (\psi(v_i v_{i-1}) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1}) + \psi(v_i u_i)) \pmod{2n-1} \\ &= \left(\frac{3n+9}{2}\right) \pmod{2n-1}.\end{aligned}$$

For $i = n, n \geq 7$,

$$\sigma_{\psi}(v_n) = (\psi(v_i v_{i-1}) + \psi(v_i u_{i-1})) \pmod{2n-1} = \left(3 + \frac{n-1}{2}\right) \pmod{2n-1}.$$

Hence, the set of modular weights of vertices $v_i, i = 1, 2, \dots, n$ is

$$\{\sigma_{\psi}(v_i) : 1 \leq i \leq n\} = \{0, 1, 2, 5, \dots, n-1\}.$$

Then, we evaluate the weights of vertices u_i .

For $i = 1, 2$,

$$\sigma_{\psi}(u_i) = (\psi(u_i v_i) + \psi(u_i v_{i+1})) \pmod{2n-1} = (i+2) \pmod{2n-1}.$$

For $3 \leq i \leq \frac{n-3}{2}, n \geq 9$,

$$\sigma_{\psi}(u_i) = (\psi(u_i v_i) + \psi(u_i v_{i+1})) \pmod{2n-1} = \left(\frac{n-2i+9}{2} \right) \pmod{2n-1}.$$

For $i = 3, 4, n = 5$,

$$\sigma_{\psi}(u_i) = (\psi(u_i v_i) + \psi(u_i v_{i+1})) \pmod{2n-1} = (4+i) \pmod{2n-1}.$$

For $\frac{n-1}{2} \leq i \leq n-1, n \geq 7$,

$$\sigma_{\psi}(u_i) = (\psi(u_i v_i) + \psi(u_i v_{i+1})) \pmod{2n-1} = (i+4) \pmod{2n-1}.$$

Hence, the set of modular weights of vertices $u_i, i = 1, 2, \dots, n-1$ is

$$\{\sigma_{\psi}(u_i) : 1 \leq i \leq n-1\} = \{3, 4, n-3, \dots, n+3\}.$$

So,

$$\begin{aligned} \{\sigma_{\psi}(e) : e \in V(\mathcal{TS}_n)\} &= \{\sigma_{\psi}(v_i) : 1 \leq i \leq n\} \cup \{\sigma_{\psi}(u_i) : 1 \leq i \leq n-1\} \\ &= \{0, 1, 2, 5, \dots, n-1\} \cup \{3, 4, n-3, \dots, n+3\} \\ &= \{0, 1, 2, \dots, 2n-2\}. \end{aligned}$$

Under the labeling ψ , the edge labels do not exceed $\lfloor \frac{n+2}{2} \rfloor + 1$, and the vertex weights take consecutive values $3, 4, \dots, 2n+1$. Consequently, the modular weights form the set $\{0, 1, 2, \dots, 2n-2\}$, ensuring that ψ constitutes a modular irregular labeling with the required properties.

□

Fig. 2 illustrates a specific instance of the modular irregular labeling for the triangular snake graph $(\mathcal{TS}_n)$, as described in Theorem 2.2.

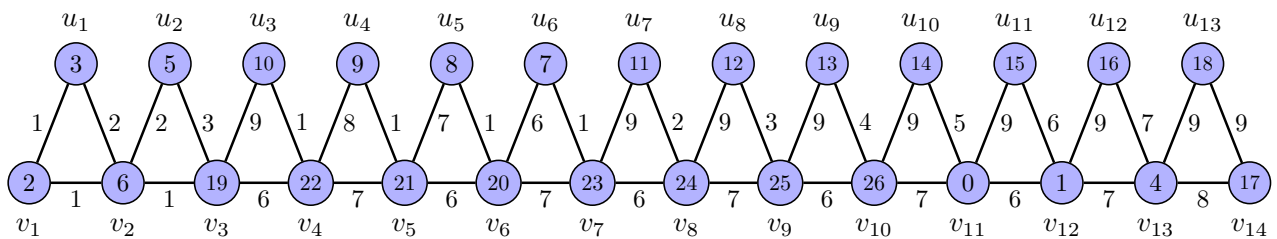


Figure 2. Irregular Labeling of Graphs $\mathcal{TS}_{14}$

4. Modular irregularity strength of middle graph of diamond graphs

We now turn our attention to the class of diamond graphs. Unlike the middle graph of a path and the triangular snake graph discussed in the previous sections, a diamond graph is constructed by a

sequence of edge-sharing diamonds, resulting in a denser local structure and a larger number of adjacent vertices. This structural difference leads to distinct labeling constraints when determining the modular irregularity strength. To establish the value of $ms(\mathcal{D}_n)$, we first consider the smallest nontrivial diamond graph and determine its exact modular irregularity strength.

Lemma 4.1. *Let $\mathcal{D}_n$ be a diamond graph with $n = 4$. It follows that $ms(\mathcal{D}_4) = 3$.*

Proof. Given that $|V(\mathcal{D}_4)| = 2n = 2(4) = 8$, assume that the graph $\mathcal{D}_4$ has a modular irregular edge-2 labeling ψ with $\psi : E(\mathcal{D}_4) \rightarrow \{1, 2\}$. Consequently, the maximum edge label value in $\mathcal{D}_4$ is 2. To obtain an edge labeling that achieves the modular irregular strength of $\mathcal{D}_4$, the labeling begins with the edges incident to the vertices v_1, u_1 and u_3 , followed by the vertex v_4 , and finally the vertex x .

The degrees of the vertices v_1, u_1, u_3 , and v_4 are three. Thus, the minimum possible weight of these vertices is 3, which is obtained by summing three edge labels, all of which have a value of 1. Suppose vertex v_1 has the minimum weight, which is 3. Therefore, vertex u_1 has a weight of 4, obtained by summing three edge labels with values 1, 1, 1, and 2. Vertex u_3 has a weight of 5, obtained by summing three edge labels with values 1, 2, and 2. Finally, the maximum weight is found at vertex v_4 , which is 6, obtained by summing three edge labels with values 2, 2, and 2.

Next, edge labeling is applied to the edges incident to vertex u_2 . Given that the degree of vertex u_2 is 4, the minimum possible weight of vertex u_2 is 7, obtained by summing four edge labels with values 1, 2, 2, and 2. Meanwhile, the maximum weight of vertex u_2 is 8, obtained by summing four edge labels with values 2, 2, 2, and 2.

Subsequently, edge labeling is applied to the edges incident to vertex x , whose degree is 4.

- (i). **Case 1:** If the weight of vertex u_2 is 7, then the weight of vertex x is also 7, obtained by summing four edge labels with values 1, 2, 2, and 2. Since the weight of vertex x is equal to that of vertex u_2 , this contradicts the definition of modular irregular labeling, which states that the weight of each vertex in graph $\mathcal{D}_4$ must be unique. Therefore, the edge labels in graph $\mathcal{D}_4$ must be at least 3.
- (ii). **Case 2:** if the weight of vertex u_2 is $8 \pmod{8} = 0$, then the weight of vertex x is 7, obtained by summing four edge labels with values 1, 2, 2, and 2. Furthermore, edge labeling is applied to the edges incident to vertices v_2 and v_3 , both of which have a degree of 5. Since the weight of vertex x is 7, the weight of vertex v_2 is $8 \pmod{8} = 0$, obtained by summing five edge labels with values 1, 1, 2, 2, and 2. Next, the weight of vertex v_3 is $9 \pmod{8} = 1$, obtained by summing five edge labels with values 1, 2, 2, 2, and 2. It follows that the weight of vertex v_3 is the same as that of vertex u_2 , which contradicts the definition of modular irregular labeling, stating that the weight of each vertex in graph $\mathcal{D}_4$ must be unique. Therefore, the edge labels in graph $\mathcal{D}_4$ must be at least 3.

Since all possible weights of vertex u_2 lead to a contradiction with the definition of modular irregular labeling, which states that the weight of each vertex $u \in V(\mathcal{D}_4)$ must be distinct, it can be concluded that a modular irregular labeling of graph $\mathcal{D}_4$ can only be achieved if the edge labels are at least 3. In other words, $ms(\mathcal{D}_4) \geq 3$.

Fig. 3 represents a 3-modular irregular labeling of the diamond graph $\mathcal{D}_4$. Thus, it can be concluded that $ms(\mathcal{D}_4) \leq 3$. Consequently, based on the trichotomy of integers, it is proven that the modular irregularity strength of the diamond graph $\mathcal{D}_4$ is $ms(\mathcal{D}_4) = 3$. □

Lemma 4.2. *Let $\mathcal{D}_n$ be a diamond graph with $n \geq 3$. If n is odd, then*

$$ms(\mathcal{D}_n) = \infty.$$

Proof. If n is an odd number, it can be written as $n = 2k + 1$ for some $k \in \mathbb{Z}$, leading to:

$$|V(\mathcal{D}_n)| = 2n = 2(2k + 1) = 4k + 2 \equiv 2 \pmod{4}.$$

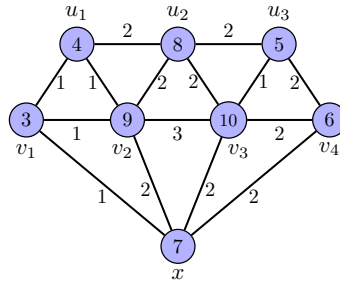


Figure 3. Irregular Labeling of Graphs $\mathcal{D}_4$

By [7], a graph with $|V(\Gamma)| \equiv 2 \pmod{4}$ does not admit a modular irregular labeling. Hence, for any odd n , the diamond graph $\mathcal{D}_n$ satisfies $ms(\mathcal{D}_n) = \infty$. $\square$

Lemma 4.3. Let $\mathcal{D}_n$ be a diamond graph with $n \geq 6$. Then

$$ms(\mathcal{D}_n) \geq \begin{cases} 2 & , \text{ if } n \leq 8, \\ \frac{n}{4} & , \text{ if } n > 8. \end{cases}$$

Proof. By Theorem 1.1, it follows that,

$$\begin{aligned} s(\mathcal{D}_n) &\geq \max_{1 \leq i \leq \Delta(\mathcal{D}_n)} \left\{ \frac{p_i - 1}{i} + 1 \right\} \\ s(\mathcal{D}_n) &\geq \max \left\{ \frac{4-1}{3} + 1, \frac{(n-3)-1}{4} + 1, \frac{(n-2)-1}{5} + 1, \frac{1-1}{n} + 1 \right\} \\ s(\mathcal{D}_n) &\geq \max \left\{ \frac{3}{3} + 1, \frac{n-4}{4} + \frac{4}{4}, \frac{n-3}{5} + \frac{5}{5}, 0 + 1 \right\} \\ s(\mathcal{D}_n) &\geq \max \left\{ 2, \frac{n}{4}, \frac{n+2}{5}, 1 \right\} \\ s(\mathcal{D}_n) &\geq \max \left\{ 2, \frac{n}{4}, \frac{n+2}{5} \right\} \end{aligned}$$

It is noted that the irregularity strength of a graph $\mathcal{D}_n$, represented as $s(\mathcal{D}_n)$, is a positive integer. Therefore, it follows that

$$s(\mathcal{D}_n) = \begin{cases} 2 & , \text{ if } 6 \leq n \leq 8, \\ \lceil \frac{n}{4} \rceil & , \text{ if } n > 8. \end{cases}$$

According to Theorem 1.1, we obtain:

$$\begin{cases} 2 & , \text{ if } 6 \leq n \leq 8, \\ \lceil \frac{n}{4} \rceil & , \text{ if } n > 8. \end{cases}$$

It has been proven that the lower bound of the modular irregularity strength of the diamond graph $\mathcal{D}_n$ holds for $n \geq 6$, then

$$ms(\mathcal{D}_n) \geq \begin{cases} 2 & , \text{ if } 6 \leq n \leq 8, \\ \lceil \frac{n}{4} \rceil & , \text{ if } n > 8. \end{cases}$$

$\square$

Furthermore, the following theorem provides the upper bound for the modular irregularity strength of the diamond graph $\mathcal{D}_n$.

Theorem 4.4. *Let $\mathcal{D}_n$ be a diamond graph with $n \geq 6$ and n is even, then*

$$ms(\mathcal{D}_n) \leq \begin{cases} \frac{n}{2} + 1 & , \text{ if } 6 \leq n \leq 8, \\ n - 3 & , \text{ if } n > 8, \end{cases}$$

Proof. Let $V(\mathcal{D}_n) = \{x\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n-1\}$ and $E(\mathcal{D}_n) = \{u_i u_{i+1} : 1 \leq i \leq n-2\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n-1\} \cup \{u_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i x : 1 \leq i \leq n\}$ be the vertex set and edge set of $\mathcal{D}_n$ respectively. The proof will be divided into 2 cases. For $n \geq 6$, n is even, we construct the following edge labeling ψ .

$$\begin{aligned} \psi(u_i u_{i+1}) &= \begin{cases} 1 & , \text{ if } i = 1, \\ \frac{n}{2} & , \text{ if } i = 2, \\ \frac{n}{2} & , \text{ if } i = 3, 4, n = 6, \\ 4 - 3(\lceil \frac{i}{2} \rceil - \lfloor \frac{i}{2} \rfloor) & , \text{ if } 3 \leq i \leq n-2, n = 8, \\ (1 + \lfloor \frac{i}{2} \rfloor - \lfloor \frac{i-1}{2} \rfloor) & , \text{ if } 3 \leq i \leq 2\lfloor \frac{n-10}{4} \rfloor + 4 \\ + (\lfloor \frac{n-2}{4} \rfloor - \lfloor \frac{n}{4} \rfloor + 2 - \lfloor \frac{i}{2} \rfloor + \lfloor \frac{i-1}{2} \rfloor) & n \geq 10, \\ \left(\left\lfloor \frac{i - (2\lfloor \frac{n-10}{4} \rfloor + 4)}{2\lfloor \frac{n-10}{4} \rfloor + 4} + 1 \right\rfloor \right) & \\ 2 + \left\lfloor \frac{i - (2\lfloor \frac{n-10}{4} \rfloor + 5)}{2} \right\rfloor - \left\lfloor \frac{i - (2\lfloor \frac{n-10}{4} \rfloor + 5)}{2} \right\rfloor & , \text{ if } 2\lfloor \frac{n-10}{4} \rfloor + 5 \leq i \leq n-2 \\ & n \geq 10, \end{cases} \\ \psi(u_i u_{i-1}) &= \begin{cases} 1 & , \text{ if } i = 2, \\ \frac{n}{2} & , \text{ if } i = 3, \\ \frac{n}{2} & , \text{ if } i = 4, 5, n = 6, \\ 1 + 3(\lfloor \frac{i+1}{2} \rfloor - \lfloor \frac{i}{2} \rfloor) & , \text{ if } 4 \leq i \leq n-1, n = 8, \\ (1 + \lceil \frac{i}{2} \rceil - \lfloor \frac{i}{2} \rfloor) & , \text{ if } 4 \leq i \leq 2\lfloor \frac{n-10}{4} \rfloor + 5 \\ + (\lfloor \frac{n-2}{4} \rfloor - \lfloor \frac{n}{4} \rfloor + 2 - \lceil \frac{i}{2} \rceil + \lfloor \frac{i}{2} \rfloor) & n \geq 10, \\ \left(\left\lfloor \frac{i - (2\lfloor \frac{n-10}{4} \rfloor + 5)}{2\lfloor \frac{n-10}{4} \rfloor + 5} + 1 \right\rfloor \right) & \\ 2 + \left\lfloor \frac{i - (2\lfloor \frac{n-10}{4} \rfloor + 6)}{2} \right\rfloor - \left\lfloor \frac{i - (2\lfloor \frac{n-10}{4} \rfloor + 6)}{2} \right\rfloor & , \text{ if } 2\lfloor \frac{n-10}{4} \rfloor + 6 \leq i \leq n-1 \\ & n \geq 10, \end{cases} \\ \psi(u_i v_i) &= \begin{cases} \frac{n}{2} - 1 & , \text{ if } i = 1, \\ \frac{n}{2} & , \text{ if } i = 2, \\ \frac{n}{2} & , \text{ if } i = 3, n = 4, \\ 3(1 - (\lceil \frac{i}{2} \rceil - \lfloor \frac{i}{2} \rfloor)) + (2 + 2\lfloor \frac{i-3}{2} \rfloor) & , \text{ if } 2 \leq i \leq n-1, n = 6, \\ (\lceil \frac{i}{2} \rceil - \lfloor \frac{i}{2} \rfloor) & \\ n - 3 & , \text{ if } 3 \leq i \leq n-1, n \geq 8, \end{cases} \end{aligned}$$

$$\psi(u_i v_{i+1}) = \begin{cases} \frac{n-2}{2} & , \text{ if } i = 1, \\ \frac{n}{2} & , \text{ if } i = 2, 3, n = 4, \\ \frac{n-2}{2} & , \text{ if } 2 \leq i \leq n-2, n = 6 \\ i - 1 & , \text{ if } 2 \leq i \leq n-2, n \geq 8, \\ \frac{n-4}{2} & , \text{ if } i = n-1, n \geq 6, \end{cases}$$

$$\begin{aligned}\psi(v_i u_{i-1}) &= \begin{cases} \frac{n-2}{2} & , \text{ if } i = 2, \\ \frac{n}{2} & , \text{ if } i = 3, 4, n = 4, \\ \frac{n-2}{2} & , \text{ if } 3 \leq i \leq n-1, n = 6, \\ i-2 & , \text{ if } 3 \leq i \leq n-1, n \geq 8, \\ \frac{n-4}{2} & , \text{ if } i = n, n \geq 6, \end{cases} \\ \psi(v_i v_{i+1}) &= \begin{cases} \frac{n-2}{2} & , \text{ if } i = 1, \\ \frac{n+2}{2} & , \text{ if } i = 2, n = 4, \\ 1 & , \text{ if } i = 3, n = 4, \\ \frac{n+2}{2} & , \text{ if } 2 \leq i \leq n-1, n \geq 6, \end{cases} \\ \psi(v_i v_{i-1}) &= \begin{cases} \frac{n-2}{2} & , \text{ if } i = 2, \\ \frac{n+2}{2} & , \text{ if } i = 3, n = 4, \\ 1 & , \text{ if } i = 4, n = 4, \\ \frac{n+2}{2} & , \text{ if } 3 \leq i \leq n, n \geq 6, \end{cases} \\ \psi(v_i x) &= 2 & , \text{ if } 1 \leq i \leq n. \end{aligned}$$

Based on the equations above, the proof proceeds by considering two distinct cases.

(i). **Case i**, For $6 \leq n \leq 8$ we obtain the following

$$\begin{aligned}\max\{\psi(u_i u_{i+1}) : 1 \leq i \leq n-1\} &= \frac{n}{2} \leq \frac{n}{2} + 1, \\ \max\{\psi(u_i u_{i-1}) : 2 \leq i \leq n-1\} &= \frac{n}{2} \leq \frac{n}{2} + 1, \\ \max\{\psi(u_i v_i) : 1 \leq i \leq n-1\} &= \frac{n}{2} + 1, \\ \max\{\psi(u_i v_{i+1}) : 1 \leq i \leq n-1\} &= \frac{n}{2} + 1, \\ \max\{\psi(v_i u_{i-1}) : 2 \leq i \leq n\} &= \frac{n}{2} + 1, \\ \max\{\psi(v_i v_{i+1}) : 1 \leq i \leq n-1\} &= \frac{n}{2} + 1, \\ \max\{\psi(v_i v_{i-1}) : 2 \leq i \leq n\} &= \frac{n}{2} + 1, \\ \max\{\psi(v_i x) : 1 \leq i \leq n\} &= 2 \leq \frac{n}{2} + 1. \end{aligned}$$

We get that ψ is an $(\frac{n}{2} + 1)$ -labeling. Next, the vertex weights are examined. First, we evaluate the weights of vertices u_i .

For $i = 1$,

$$\sigma_{\psi}(u_1) = (\psi(u_i v_i) + \psi(u_i v_{i+1}) + \psi(u_i u_{i+1})) \pmod{2n} = (n-1) \pmod{2n}.$$

For $i = 2, n = 4$,

$$\sigma_{\psi}(u_2) = (\psi(u_i u_{i-1}) + \psi(u_i v_i) + \psi(u_i v_{i+1}) + \psi(u_i u_{i+1})) \pmod{2n} = 7 \pmod{8}.$$

For $i = 2, n = 6$,

$$\sigma_{\psi}(u_2) = (\psi(u_i u_{i-1}) + \psi(u_i v_i) + \psi(u_i v_{i+1}) + \psi(u_i u_{i+1})) \pmod{2n} = 9 \pmod{12}.$$

For $i = 2, n = 8$,

$$\sigma_{\psi}(u_2) = (\psi(u_i u_{i-1}) + \psi(u_i v_i) + \psi(u_i v_{i+1}) + \psi(u_i u_{i+1})) \pmod{2n} = 10 \pmod{16}.$$

For $i = 3, 4, n = 6$,

$$\begin{aligned} \sigma_{\psi}(u_i) &= (\psi(u_i u_{i-1}) + \psi(u_i v_i) + \psi(u_i v_{i+1}) + \psi(u_i u_{i+1})) \pmod{2n} \\ &= \left(8 + 3 \left(1 - \left(\left\lceil \frac{i}{2} \right\rceil - \left\lfloor \frac{i}{2} \right\rfloor \right) \right) \right) + \left(2 + 2 \left\lfloor \frac{i-3}{2} \right\rfloor \right) \left(\left\lceil \frac{i}{2} \right\rceil - \left\lfloor \frac{i}{2} \right\rfloor \right) \pmod{2n}. \end{aligned}$$

For $i = 3, n = 8$,

$$\sigma_{\psi}(u_3) = (\psi(u_i u_{i-1}) + \psi(u_i v_i) + \psi(u_i v_{i+1}) + \psi(u_i u_{i+1})) \pmod{2n} = 12 \pmod{16}.$$

For $4 \leq i \leq n-2, n = 8$,

$$\sigma_{\psi}(u_i) = (\psi(u_i u_{i-1}) + \psi(u_i v_i) + \psi(u_i v_{i+1}) + \psi(u_i u_{i+1})) \pmod{2n} = (i+9) \pmod{16}.$$

For $i = n-1, n = 4$,

$$\sigma_{\psi}(u_{n-1}) = (\psi(u_i u_{i-1}) + \psi(u_i v_i) + \psi(u_i v_{i+1})) \pmod{2n} = 6 \pmod{8}.$$

For $i = n-1, n = 6$,

$$\sigma_{\psi}(u_{n-1}) = (\psi(u_i u_{i-1}) + \psi(u_i v_i) + \psi(u_i v_{i+1})) \pmod{2n} = 8 \pmod{12}.$$

For $i = n-1, n = 8$,

$$\sigma_{\psi}(u_{n-1}) = (\psi(u_i u_{i-1}) + \psi(u_i v_i) + \psi(u_i v_{i+1})) \pmod{2n} = 11 \pmod{16}.$$

Hence, the set of modular weights of vertices $u_i, i = 1, 2, \dots, n-1$ is

$$\{\sigma_{\psi}(u_i) : 1 \leq i \leq n-1\} = \{3, 5, \dots, 2n-1\}.$$

Now, we evaluate the weights of vertices v_i .

For $i = 1$,

$$\sigma_{\psi}(v_i) = (\psi(v_i x) + \psi(v_i v_{i+1}) + \psi(v_i u_i)) \pmod{2n} = n \pmod{2n}.$$

For $i = 2, n = 4$,

$$\sigma_{\psi}(v_2) = (\psi(v_i v_{i-1}) + \psi(v_i x) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1})) \pmod{2n} = 9 \pmod{8}.$$

For $i = 2, n = 6, 8$,

$$\sigma_{\psi}(v_2) = (\psi(v_i v_{i-1}) + \psi(v_i x) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1})) \pmod{2n} = (2n-1) \pmod{2n}$$

For $i = 3, n = 4$

$$\sigma_{\psi}(v_3) = (\psi(v_i v_{i-1}) + \psi(v_i x) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1})) \pmod{2n} = 10 \pmod{8}.$$

For $3 \leq i \leq n-1, n = 6$,

$$\begin{aligned} \sigma_{\psi}(v_i) &= (\psi(v_i v_{i-1}) + \psi(v_i x) + \psi(v_i v_{i+1}) + \psi(v_i u_i) + \psi(v_i u_{i-1})) \pmod{2n} \\ &= \left(12 + 3 \left(1 - \left(\left\lceil \frac{i}{2} \right\rceil - \left\lfloor \frac{i}{2} \right\rfloor \right) \right) \right) + \left(2 + 2 \left\lfloor \frac{i-3}{2} \right\rfloor \right) \left(\left\lceil \frac{i}{2} \right\rceil - \left\lfloor \frac{i}{2} \right\rfloor \right) \pmod{2n}. \end{aligned}$$

For $3 \leq i \leq n-1, n = 8$

$$\sigma_{\psi}(v_i) = (\psi(v_i v_{i-1}) + \psi(v_i x) + \psi(v_i v_{i+1}) + \psi(v_i u_i) + \psi(v_i u_{i-1})) \pmod{2n} = (2n+i-1) \pmod{2n}.$$

For $i = 5, n = 4$,

$$\sigma_{\psi}(v_i) = (\psi(v_i x) + \psi(v_i v_{i-1}) + \psi(v_i u_{i-1})) \pmod{2n} = 5 \pmod{8}.$$

For $i = 5, n \geq 6$,

$$\sigma_{\psi}(v_i) = (\psi(v_i x) + \psi(v_i v_{i-1}) + \psi(v_i u_{i-1})) \pmod{2n} = (n+1) \pmod{2n}.$$

Hence, the set of modular weights of vertices $v_i, i = 1, 2, \dots, n$ is.

$$\{\sigma_{\psi}(v_i) : 1 \leq i \leq n\} = \{1, 2, \dots, n-2\}.$$

Furthermore, for x we have

$$\sigma_{\psi}(x) = \sum_{i=1}^n (\psi(v_i x)) \pmod{2n} = \sum_{i=1}^n 2 \pmod{2n} = (2n) \pmod{2n} = 0.$$

Therefore,

$$\begin{aligned} \{\sigma_{\psi}(w) : w \in V(\mathcal{D}_n)\} &= \{\sigma_{\psi}(v_i) : 1 \leq i \leq n\} \cup \{\sigma_{\psi}(u_i) : 1 \leq i \leq n-1\} \cup \{\sigma_{\psi}(x)\} \\ &= \{3, 5, \dots, 2n-1\} \cup \{1, 2, \dots, n-2\} \cup \{0\} \\ &= \{0, 1, \dots, 2n-1\}. \end{aligned}$$

Hence, it is established that, for even n with $6 \leq n \leq 8$, all vertex modular weights are distinct.

(ii). **Case ii**, For $n > 8$ we obtain the following

$$\begin{aligned} \max\{\psi(u_i u_{i+1}) : 1 \leq i \leq n-1\} &= \frac{n}{2} \leq n-3, \\ \max\{\psi(u_i u_{i-1}) : 2 \leq i \leq n-1\} &= \frac{n}{2} \leq n-3, \\ \max\{\psi(u_i v_i) : 1 \leq i \leq n-1\} &= n-3, \\ \max\{\psi(u_i v_{i+1}) : 1 \leq i \leq n-1\} &= n-3, \\ \max\{\psi(v_i u_{i-1}) : 2 \leq i \leq n\} &= n-3, \\ \max\{\psi(v_i v_{i+1}) : 1 \leq i \leq n-1\} &= \frac{n}{2} + 1 \leq n-3, \\ \max\{\psi(v_i v_{i-1}) : 2 \leq i \leq n\} &= \frac{n}{2} + 1 \leq n-3, \\ \max\{\psi(v_i x) : 1 \leq i \leq n\} &= 2 \leq n-3. \end{aligned}$$

We get that ψ is an $(n-3)$ -labeling. Next, the vertex weights are examined. First, we evaluate the weights of vertices u_i

For $i = 1$,

$$\sigma_{\psi}(u_1) = (\psi(u_i v_i) + \psi(u_i v_{i+1}) + \psi(u_i u_{i+1})) \pmod{2n} = (n-1) \pmod{2n}.$$

For $i = 2, n > 8$,

$$\sigma_{\psi}(u_2) = (\psi(u_i u_{i-1}) + \psi(u_i v_i) + \psi(u_i v_{i+1}) + \psi(u_i u_{i+1})) \pmod{2n} = (n+2) \pmod{2n}.$$

For $i = 3, n \geq 10$,

$$\sigma_{\psi}(u_3) = (\psi(u_i u_{i-1}) + \psi(u_i v_i) + \psi(u_i v_{i+1}) + \psi(u_i u_{i+1})) \pmod{2n} = \left(\frac{3n}{2}\right) \pmod{2n}.$$

For $4 \leq i \leq \frac{n}{2} - 2, n \geq 12$,

$$\sigma_{\psi}(u_3) = (\psi(u_i u_{i-1}) + \psi(u_i v_i) + \psi(u_i v_{i+1}) + \psi(u_i u_{i+1})) \pmod{2n} = (n+i-1) \pmod{2n}.$$

For $i = \frac{n}{2} - 1, n \geq 10$,

$$\sigma_{\psi}(u_i) = (\psi(u_i u_{i-1}) + \psi(u_i v_i) + \psi(u_i v_{i+1})) \pmod{2n} = (n+i) \pmod{2n}.$$

For $\frac{n}{2} \leq i \leq n-2, n \geq 10$,

$$\sigma_{\psi}(u_i) = (\psi(u_i u_{i-1}) + \psi(u_i v_i) + \psi(u_i v_{i+1})) \pmod{2n} = (n + i + 1) \pmod{2n}.$$

For $i = n - 1, n \geq 10$,

$$\sigma_{\psi}(u_{n-1}) = (\psi(u_i u_{i-1}) + \psi(u_i v_i) + \psi(u_i v_{i+1})) \pmod{2n} = \left(\frac{3n}{2} - 2\right) \pmod{2n}.$$

Hence, the set of modular weights of vertices $u_i, i = 1, 2, \dots, n - 1$ is

$$\{\sigma_{\psi}(u_i) : 1 \leq i \leq n - 1\} = \{9, 11, \dots, 2n - 1\}.$$

Then, we evaluate the weights of vertices v_i

For $i = 1$,

$$\sigma_{\psi}(v_i) = (\psi(v_i x) + \psi(v_i v_{i+1}) + \psi(v_i u_i)) \pmod{2n} = (n) \pmod{2n}.$$

For $i = 2, n > 8$,

$$\sigma_{\psi}(v_2) = (\psi(v_i v_{i-1}) + \psi(v_i x) + \psi(v_i v_{i+1}) + \psi(v_i u_{i-1})) \pmod{2n} = (2n - 1) \pmod{2n}.$$

For $3 \leq i \leq n - 1, n > 8$

$$\begin{aligned} \sigma_{\psi}(v_i) &= (\psi(v_i v_{i-1}) + \psi(v_i x) + \psi(v_i v_{i+1}) + \psi(v_i u_i) + \psi(v_i u_{i-1})) \pmod{2n} \\ &= (2n + i - 1) \pmod{2n}. \end{aligned}$$

For $i = n - 1, n > 8$,

$$\sigma_{\psi}(v_i) = (\psi(v_i x) + \psi(v_i v_{i-1}) + \psi(v_i u_{i-1})) \pmod{2n} = (n + 1) \pmod{2n}$$

Hence, the set of modular weights of vertices $v_i, i = 1, 2, \dots, n$ is

$$\{\sigma_{\psi}(v_i) : 1 \leq i \leq n\} = \{2, 3, \dots, n + 1\}.$$

The weights of vertex x is

$$\sigma_{\psi}(x) = \sum_{i=1}^n (\psi(v_i x)) \pmod{2n} = \sum_{i=1}^n 2 \pmod{2n} = (2n) \pmod{2n} = 0 \pmod{2n}.$$

Thus,

$$\begin{aligned} \{\sigma_{\psi}(w) : w \in V(\mathcal{D}_n)\} &= \{\sigma_{\psi}(v_i) : 1 \leq i \leq n\} \cup \{\sigma_{\psi}(u_i) : 1 \leq i \leq n - 1\} \cup \{\sigma_{\psi}(x)\} \\ &= \{9, 11, \dots, 2n - 1\} \cup \{2, 3, \dots, n + 1\} \cup \{0\} \\ &= \{0, 1, \dots, 2n - 1\}. \end{aligned}$$

Hence, for even n with $n > 8$, all vertex modular weights are distinct. □

5. Conclusion

In this paper, we obtained the exact modular irregularity strength for the middle graphs of paths and the triangular snake graphs. For the diamond graphs, we showed that the modular irregularity strength is infinite whenever n is odd, determined the exact value for $\mathcal{D}_4$, and derived lower and upper bounds for even orders $n \geq 6$. These findings enrich the theory of modular irregular labelings and further illustrate how graph structure affects modular irregularity strength.

Despite these advances, determining the exact value of $ms(\mathcal{D}_n)$ for even $n \geq 6$ remains an open problem. More generally, the modular irregularity strength of many graph families is still unknown. We conclude the paper with an open problem for future research.

Problem 1. Determine the modular irregularity of diamond graph $\mathcal{D}_n$, where $n \geq 6$, and n is even.

Acknowledgment

The authors are grateful to the reviewers for their invaluable comments and constructive suggestions.

Declarations

Author contributions

All authors contributed to the manuscript and approved the final version.

Conflict of interest

The author(s) declare that they have no conflict of interest.

Data availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Funding

The author(s) declare that no funds, grants, or other support were received during the preparation of this manuscript.

Generative AI and AI-assisted technologies

The author(s) declare that no generative AI or AI-assisted technologies were used in the preparation of this manuscript, except for tools used solely for spelling, grammar, language editing, reference formatting, or similar routine editorial corrections.

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