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Skew cyclic codes over a finite non-chain ring and an application

Research Article

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Abstract: This article studies Θ_t-cyclic and (Θ_t, λ)-constacyclic codes over the finite commutative non-chain Frobenius ring $R = \mathbb{F}_q[u, v, w]/\langle u^2-u,\ v^2-v,\ w^2-1,\ uv,\ uw-wu,\ wv-vw\rangle$. Gray maps, structural decompositions, and generator descriptions are developed for both odd- and even-characteristic cases. The paper further determines principal generators in the associated skew polynomial rings, dual codes, idempotent generators, and conditions for self-duality. It also presents explicit examples over specific finite fields and extends the framework to DNA codes in the even-characteristic setting through reversibility and complement constraints. Spanning sets, cardinality formulas, and optimal DNA-code constructions meeting the Griesmer bound are also obtained.

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1. Introduction

In 2007, Boucher et al. [9] used the skew polynomials ring $\mathbb{F}[x;\theta]$ into coding theory where θ is an automorphism on the finite field $\mathbb{F}$. The ring $\mathbb{F}[x;\theta]$ is a non-commutative as coefficients in $\mathbb{F}$ and indeterminate x are not commuting. They introduced skew cyclic (or θ-cyclic), a generalized class of cyclic

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codes as a principally generated ideal of the ring $\mathbb{F}[x;\theta]/\langle x^n-1\rangle$, where the order of the automorphism divides n and derived many new codes that were not previously obtained. One advantage of studying codes over skew polynomial rings is that the polynomial x^n-1 has more factors in skew polynomial ring than commutative rings. Later, in [10] skew constacyclic codes that generalize the constacyclic codes over commutative rings are studied. In 2011, Siap et al. [30] discussed skew cyclic codes of arbitrary length and identified them as a left $\mathbb{F}[x;\theta]$-submodule of $\mathbb{F}[x;\theta]/\langle x^n-1\rangle$. In 2012, Abualrub et al. [2] studied θ-cyclic codes over the non-chain ring $\mathbb{F}_2+u\mathbb{F}_2, v^2=v$, and Jitman et al. [21] studied skew constacyclic codes over finite chain ring. Later, these codes over non-chain rings are extensively studied. For instance, the rings $\mathbb{F}_3+v\mathbb{F}_3$ in [5], $\mathbb{F}_q+v\mathbb{F}_q, v^2=v$ in [18], $\mathbb{F}_q+u\mathbb{F}_q+v\mathbb{F}_q, u^2=u, v^2=v, uv=vu=0$ in [6] are considered to study skew cyclic codes. Also, Yao et al. [32] and Dertli and Cengellenmis [12] studied these codes over $\mathbb{F}_q+u\mathbb{F}_q+v\mathbb{F}_q+uv\mathbb{F}_q, u^2=u, v^2=v, uv=vu$. In [17], the studies on constacyclic and skew constacyclic codes over non-chain rings have received considerable attention in recent years. In particular, Gowdhaman et al. studied constacyclic codes over the non-chain finite commutative ring $\mathbb{Z}_4+u\mathbb{Z}_4+v\mathbb{Z}_4$, establishing structural properties and applications. Furthermore, [25] Mohan et al. investigated skew-constacyclic codes over non-chain rings, providing important insights into their algebraic structure and generating techniques. These works motivate the present study on Θ_t-cyclic and (Θ_t,λ)-cyclic codes over a more general non-chain ring. In 2017, Gao et al. [16] obtained the structure of skew constacyclic codes over non-chain ring $\mathbb{F}_q+v\mathbb{F}_q, v^2=v$. Recently, Islam and Prakash [19, 20] have determined structural properties of skew constacyclic codes over $\mathbb{F}_q+u\mathbb{F}_q+v\mathbb{F}_q+uv\mathbb{F}_q, u^2=u, v^2=v, uv=vu$ and $\mathbb{F}_q+u\mathbb{F}_q+v\mathbb{F}_q, u^2=u, v^2=v, uv=vu=0$, respectively. In 2019, Bhardwaj and Raka [8] studied skew constacyclic codes over the ring $\mathbb{F}_q[u,v]/\langle f(u), g(v), uv-vu\rangle$.

Motivated by the studies above, here we consider a non-chain Frobenius ring $R=\mathbb{F}_q[u,v,w]/\langle u^2=u, v^2=v, w^2=1, uv=vu=0, uw=wu, wv=vw\rangle$ and study Θ_t-cyclic and (Θ_t,λ)-cyclic codes over it where λ is a unit in R. We completely determine the structure of these codes and obtain necessary and sufficient conditions to contain their duals. Note that the rings $\mathbb{F}_q+u\mathbb{F}_q+v\mathbb{F}_q, u^2=u, v^2=v, uv=vu=0$ and $\mathbb{F}_q+w\mathbb{F}_q, w^2=1$ are contained in R as subrings and hence, our study and findings generalize the studies mentioned above for odd characteristic case. Further we also tackle the even characteristic case and we present some applications to DNA codes.

2. Preliminaries

Here we study the structure of the ring $R=\mathbb{F}_q[u,v,w]/\langle u^2=u, v^2=v, w^2=1, uv=vu=0, uw=wu, wv=vw\rangle$. First suppose that the characteristic is odd i.e., p is an odd prime and $q=p^m$. Then R is a finite commutative, Frobenius and non-chain ring with identity and q^6 elements. The ring R has another representation $R=\mathbb{F}_q+u\mathbb{F}_q+v\mathbb{F}_q+w(\mathbb{F}_q+u\mathbb{F}_q+v\mathbb{F}_q)=\mathbb{F}_q+u\mathbb{F}_q+v\mathbb{F}_q+w\mathbb{F}_q+uw\mathbb{F}_q+vw\mathbb{F}_q$ with the condition $u^2=u, v^2=v, w^2=1, uv=vu=0, uw=wu, vw=wv$. Any element of R has the unique expression $r=a_1+ua_2+va_3+wa_4+uwa_5+vwa_6$ where $a_i\in\mathbb{F}_q$, for $1\le i\le 6$. Let

$$e_1=\frac{u(1-w)}{2}, e_2=\frac{v(1-w)}{2}, e_3=\frac{u(1+w)}{2}, e_4=\frac{v(1+w)}{2}, e_5=\frac{(1-u-v)(1-w)}{2}$$
$$\text{and } e_6=\frac{(1-u-v)(1+w)}{2}.$$

Then the $\{e_1,e_2,\ldots,e_6\}$ is a set of orthogonal idempotent elements in R, or in other words,

$$e_i^2=e_i, e_ie_j=0, \text{ when } i\neq j \text{ and } \sum_{i=1}^{6}e_i=1.$$

Therefore, the ring R has the decomposition $R=\bigoplus_{i=1}^6 e_iR$. Since $e_iR\cong e_i\mathbb{F}_q$, for $1\le i\le 6$, then

$$R\cong\bigoplus_{i=1}^{6}e_i\mathbb{F}_q.$$

Hence, $r \in R$ has a unique representation $r = \bigoplus_{i=1}^{6} e_i r_i$, where $r_i \in \mathbb{F}_q$ for $1 \leq i \leq 6$. We recall the Frobenius automorphism θ_t on $\mathbb{F}_q$ defined by $\theta_t(a) = a^{p^t}$, where $t|m$. The extension of the automorphism on R is defined as

$$\Theta_t(r) = \sum_{i=1}^{6} e_i \theta_t(r_i) = \sum_{i=1}^{6} e_i r_i^{p^t}, \text{ where } r = \sum_{i=1}^{6} e_i r_i \text{ and } r_i \in \mathbb{F}_q \text{ for } 1 \leq i \leq 6.$$

In the even characteristic case we do not have a full idempotent decomposition as finite fields but still we have a direct decomposition of chain rings.

Given an element $r = a + bu + cv \in R$ where $a, b, c \in R_w = \mathbb{F}_q[w]/\langle w^2 - 1\rangle$ we have

$$r = (a + b)u + (a + c)v + a(1 + u + v)$$

and clearly this leads to a direct idempotent decomposition of R as

$$R = R_w u \oplus R_w v \oplus R_w(1 + u + v).$$

Here, $R_w = F_{2^m}[w]/\langle w^2 - 1\rangle$ is a local ring with chain of ideals

$$\{0\} \subset \langle 1 + w\rangle \subset R_w$$

and $\langle 1 + w\rangle$ is its maximal ideal.

Now, recall the multiplication of skew polynomials which is defined by the distribution under the rule $(ax^i)(bx^j) = a\Theta_t(b)^i x^{i+j}$. Therefore, the skew polynomial ring $R[x; \Theta_t] = \{f(x) \in R[x]\}$ is a non-commutative ring under the above multiplication and standard addition of polynomials. It becomes a commutative ring whenever we considered Θ_t as an identity automorphism. Further, the polynomial $x^n - 1$ is a central element in $R[x; \Theta_t]$ if $o(\Theta_t)|n$. In that case $\langle x^n - 1\rangle$ is a two sided ideal and hence, $R_n = R[x; \Theta_t]/\langle x^n - 1\rangle$ is a ring. However, under the left multiplication defined by $c(x)(a(x) + \langle x^n - 1\rangle) = c(x)a(x) + \langle x^n - 1\rangle$, where $c(x) \in R[x; \Theta_t]$ and $a(x) + \langle x^n - 1\rangle \in R_n$, the structure is well defined. Hence R_n is a left $R[x; \Theta_t]$-module.

Definition 2.1. *A non-empty subset C of R^n is called a linear code of length n over R if C is an R-submodule of R^n and the elements of C are called codewords.*

Definition 2.2. *[9] A non-trivial R-submodule C of R^n is called a Θ_t-cyclic code if for any $c = (c_0, c_1, \ldots, c_{n-1}) \in C$, $\sigma(c) = (\Theta_t(c_{n-1}), \Theta_t(c_0), \ldots, \Theta_t(c_{n-2})) \in C$. The operator σ is called as Θ_t-cyclic shift operator on R^n.*

A well known theorem that establishes the structure of skew cyclic codes is the following:

Theorem 2.3. *Let $c(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} \in R_n$ be a polynomial corresponding to the codeword $c = (c_0, c_1, \ldots, c_{n-1}) \in C$. Then the linear code C is a Θ_t-cyclic code if and only if C is a left $R[x; \Theta_t]$-submodule of R_n.*

We recall the Euclidean inner product of any two vectors $c = (c_0, c_1, \ldots, c_{n-1}), d = (d_0, d_1, \ldots, d_{n-1})$ is $c \cdot d = \sum_{i=0}^{n-1} c_i d_i$. For a linear code C of length n, its dual code $C^{\perp} = \{c \in R^n \mid c \cdot d = 0, \text{ for all } d \in C\}$ is also a linear code of length n. Also, C is called a *self-dual* if $C = C^{\perp}$ and *self-orthogonal* if $C \subseteq C^{\perp}$. Also, it is not hard to check that $C^{\perp}$ is a θ_t-cyclic code of length n over R, when C is a θ_t-cyclic code over $\mathbb{F}_q$.

3. Gray maps and $\mathbb{F}_q$-images

Here, we propose Gray maps for odd and even characteristic cases:

If the characteristic is odd, then we define $\phi_o : R \longrightarrow \mathbb{F}_q^6$ by

$$\phi_o(r) = (r_1, r_2, \ldots, r_6) \text{ where } r = \sum_{i=1}^{6} e_i r_i \text{ and } r_i \in \mathbb{F}_q \text{ for } 1 \leq i \leq 6. \tag{1}$$

If the characteristic is even, then we define $\phi_e : R \longrightarrow R_w^3$ by

$$\phi_e(r) = (a + b, a + c, a) \text{ where } r = a + bu + cv \in R,\ a, b, c \in R_w. \tag{2}$$

Then, both maps ϕ_o and ϕ_e are linear bijective and can be extended over R^n component-wise. We define the Gray weight w_G of $c \in R$ by $w_{o_G}(c) = w_H(\phi_o(c))$ (and $w_{e_G}(c) = w_H(\phi_e(c))$) where w_H is the Hamming weight in R^n in odd (even) characteristic case. From now on, we omit the notations w_o and w_e and use w instead, as the choice will be clear from the context, i.e., the characteristic. The Gray weight of a codeword is the rational sum of weights of components and distance between two codewords $c, d \in C$ is $d_G(c, d) = w_G(c - d)$. The Gray distance for the code C is $d_G(C) = min\{w_G(c) \mid 0 \neq c \in C\}$. For $c, d \in R^n$, $d_G(c, d) = w_G(c - d) = w_H(\phi(c - d)) = w_H(\phi(c) - \phi(d)) = d_H(\phi(c), \phi(d))$. Hence ϕ is a linear distance preserving map from (R^n, d_G) to $(\mathbb{F}_q^{6n}, d_H)$ in odd case and from (R^n, d_G) to (R_w^{3n}, d_H) in even case.

Let C be a linear code of length n over R. Given the direct decomposition of the ring R, C is also a direct sum as $C = \bigoplus_{i=1}^{6} e_i C_i$ in odd case and $C = \bigoplus_{i=1}^{3} e_i C_i$ in even case where the components are from the finite field $\mathbb{F}_q$ and the quotient ring R_w respectively.

Theorem 3.1. *Let C be a linear code of length n over R. If the characteristic of R is odd, then $\phi_o(C^\perp) = (\phi_o(C))^\perp$, and if the characteristic of R is even, then $\phi_e(C^\perp) = (\phi_e(C))^\perp$.*

Proof. First assume that the characteristic is odd. Let $c = (c_0, c_1, \ldots, c_{n-1}) \in C$ and $d = (d_0, d_1, \ldots, d_{n-1}) \in C^\perp$ where $c_j = \sum_{i=1}^{6} e_i s_j^i$ and $d_j = \sum_{i=1}^{6} e_i t_j^i$ for $s_j^i, t_j^i \in \mathbb{F}_q$, $0 \leq j \leq n - 1$. Now, $c \cdot d = 0$ implies $\sum_{j=0}^{n-1} c_j d_j = 0$, i.e., $\sum_{j=0}^{n-1} \sum_{i=1}^{6} e_i s_j^i t_j^i = 0$. Again, we have

$$\phi_o(c) \cdot \phi_o(d) = \sum_{j=0}^{n-1} \sum_{i=1}^{6} s_j^i t_j^i = 0.$$

Therefore, $\phi_o(C^\perp) \subseteq (\phi_o(C))^\perp$. As ϕ_o is bijective linear map, $\mid \phi_o(C^\perp) \mid = \mid (\phi_o(C))^\perp \mid$. Therefore $\phi_o(C^\perp) = (\phi_o(C))^\perp$.

The even case also follows in a similar way so we omit its proof. □

Corollary 3.2. *Let C be a linear code of length n over R. Then C is self-dual if and only if $\phi_o(C)$ is self-dual in odd characteristic case. Further $\phi_o(C)$ is self-orthogonal over $\mathbb{F}_q$ in odd characteristic if C is self-orthogonal.*

Proof. Let C be a self-dual linear code of length n over R. That is $C = C^\perp$. Then $\phi_o(C) = \phi_o(C^\perp)$ for odd characteristic and hence by Theorem 3.1, we have $\phi_o(C) = (\phi_o(C))^\perp$. Thus $\phi_o(C)$ is a self-dual linear code of length $6n$ over $\mathbb{F}_q$. Conversely, let $\phi_o(C)$ be a self-dual linear code of length $6n$ over $\mathbb{F}_q$. Then $\phi_o(C) = (\phi_o(C))^\perp$, and hence by Theorem 3.1, we have $\phi_o(C) = \phi_o(C^\perp)$. Since ϕ_o is bijective, $C = C^\perp$. Therefore, C is a self-dual linear code of length n over R.
Moreover, let C be a self-orthogonal linear code over $\mathbb{F}_q$. In other words, $C \subseteq C^\perp$ implies, $\phi_o(C) \subseteq \phi_o(C^\perp)$. Now, by Theorem 3.1, we have $\phi_o(C) \subseteq (\phi_o(C))^\perp$. Hence $\phi_o(C)$ is a self-orthogonal linear code of length $6n$ over $\mathbb{F}_q$. □

Corollary 3.3. *Let C be a linear code of length n over R. Then C is self-dual if and only if $\phi_e(C)$ is self-dual in even characteristic case. Further $\phi_e(C)$ is self-orthogonal over $\mathbb{F}_q$ in even characteristic if C is self-orthogonal.*

Proof. The proof of the Corollary is similar to the proof of Corollary 3.2. □

Theorem 3.4. *Let C be a linear code of length n over R. If the characteristic is odd, then $\phi_o(C) = \bigotimes_{i=1}^{6} C_i$ and $\mid C \mid = \prod_{i=1}^{6} \mid C_i \mid$ where C_i are linear codes over $\mathbb{F}_q$ and else if the characteristic is even, then $\phi_e(C) = \bigotimes_{i=1}^{3} C_i$ and $\mid C \mid = \prod_{i=1}^{3} \mid C_i \mid$ where the C_i are linear codes over R_w.*

Proof. Suppose the characteristic is odd. Let $z = (a_0^1, a_1^1, \ldots, a_{n-1}^1, a_0^2, a_1^2, \ldots, a_{n-1}^2, \cdots, a_0^6, a_1^6, \ldots, a_{n-1}^6)$ $\in \phi_o(C)$ and $r_i = \sum_{j=1}^{6} e_j a_i^j$, for $0 \le i \le n-1$. The map ϕ_o being bijective, so $r = (r_0, r_1, \ldots, r_{n-1}) \in C$. Therefore, by the definition of C_i, we have $(a_0^i, a_1^i, \ldots, a_{n-1}^i) \in C_i$ for $1 \le i \le 6$. Therefore, $z \in \bigotimes_{i=1}^{6} C_i$ and hence $\phi_o(C) \subseteq \bigotimes_{i=1}^{6} C_i$.

Conversely, let $z = (a_0^1, a_1^1, \ldots, a_{n-1}^1, a_0^2, a_1^2, \ldots, a_{n-1}^2, \cdots, a_0^6, a_1^6, \ldots, a_{n-1}^6) \in \bigotimes_{i=1}^{6} C_i$. Then $a^i = (a_0^i, a_1^i, \ldots, a_{n-1}^i) \in C_i$ for $1 \le i \le 6$. To show $z \in \phi_o(C)$ in the odd-characteristic proof, we have to find $z' = \sum_{i=1}^{6} e_i a_i \in C$ such that $\phi(z') = z$. Consider $z' = \sum_{i=1}^{6} e_i a^i$ and then $\phi_o(z') = z$. Consequently, $\bigotimes_{i=1}^{6} C_i \subseteq \phi_o(C)$. Combining both sides, we have $\phi_o(C) = \bigotimes_{i=1}^{6} C_i$.
Moreover, since ϕ_o is a bijection, $\mid C \mid = \mid \phi_o(C) \mid$. Consequently, $\mid C \mid = \mid \bigotimes_{i=1}^{6} C_i \mid = \prod_{i=1}^{6} \mid C_i \mid$. The even case follows in a similar way. □

Corollary 3.5. *Suppose the characteristic is odd. Let $C = \bigoplus_{i=1}^{6} e_i C_i$ be a linear code of length n over R. Then the generator matrix for C is $M = \begin{pmatrix} e_1 M_1 \\ e_2 M_2 \\ \vdots \\ e_6 M_6 \end{pmatrix}$, where M_i's $(i = 1, 2, \ldots, 6)$ are generator matrices of the C_i's $(i = 1, 2, \ldots, 6)$ over F_q, respectively. And if the characteristic is even, then the generator matrix also can be put into the following form $N = \begin{pmatrix} uN_1 \\ vN_2 \\ (1+u+v)N_3 \end{pmatrix}$, where N_i's $(i = 1, 2, 3)$ are generator matrices of the C_i's $(i = 1, 2, 3)$ R_w, respectively.*

Corollary 3.6. *Let $C = \bigoplus_{i=1}^{6} e_i C_i$ $(C = uC_1 \oplus vC_2 \oplus \bar{u}vC_3)$ be a linear code of length n over R of odd (even) characteristic. Then $\phi_o(C)$ $(\phi_e(C))$ is a $[6n, \sum_{i=1}^{6} k_i, d_H(C)]$ $([3n, \sum_{i=1}^{3} k_i, d_H(C)])$, where each C_i is an $[n, k_i, d_H(C_i)]$ linear code over $\mathbb{F}_q$ (R_w) for $1 \le i \le 6$ (3) and $d_H(C) = min\{d_H(C_i) \mid i = 1, 2, \ldots, 6\}$ $(d_H(C) = min\{d_H(C_i) \mid i = 1, 2, 3\})$.*

Theorem 3.7. *Let $C = \bigoplus_{i=1}^{6} e_i C_i$ $(C = uC_1 \oplus vC_2 \oplus (1+u+v)vC_3)$ be a linear code of length n over R of odd (even) characteristic. Then $C^\perp = \bigoplus_{i=1}^{6} e_i C_i^\perp$ $(C^\perp = uC_1\perp \oplus vC_2\perp \oplus \bar{u}vC_3^\perp)$. Moreover, C is self-dual code if and only if C_i's $(i = 1, 2, \ldots, 6)$ $(C_i'$s $(i = 1, 2, 3))$ are self-dual codes over $\mathbb{F}_q$ (R_w).*

Proof. Suppose the characteristic is odd. Let
$\overline{C}_i = \{r_i \in \mathbb{F}_q^n \mid$ there exists $r_1, r_2, \ldots, r_{i-1}, r_{i+1}, \ldots, r_6 \in \mathbb{F}_q^n$ such that $\sum_{i=1}^{6} e_i r_i \in C^\perp\}$. Then $C^\perp$ has the unique expression $C^\perp = \bigoplus_{i=1}^{6} e_i \overline{C}_i$. It is easy to see $\overline{C}_1 \subseteq C_1^\perp$. If $z \in C_1^\perp$, then $z \cdot x_1 = 0$ for all $x_1 \in C_1$. Let $s = \sum_{i=1}^{6} e_i x_i \in C$. Then $e_1 zs = e_1 x_1 z = 0$, and which implies $e_1 z \in C^\perp$. From the construction of $C^\perp$, we have $z \in \overline{C}_1$. Therefore, $C_1^\perp \subseteq \overline{C}_1$. Hence, $\overline{C}_1 = C_1^\perp$. By a similar process we have $C_i^\perp = \overline{C}_i$ for $i = 2, \ldots, 6$. Consequently, $C^\perp = \bigoplus_{i=1}^{6} e_i C_i^\perp$.
Moreover, let C be a self-dual linear code. Then $C = C^\perp$, i.e., $\bigoplus_{i=1}^{6} e_i C_i = \bigoplus_{i=1}^{6} e_i C_i^\perp$, Hence $C_i^\perp = C_i$ for $1 \le i \le 6$. Conversely, let each C_i, for $i = 1, 2 \ldots, 6$ be a self-dual linear code. Then $C_i^\perp = C_i$ for

$1 \le i \le 6$. Thus, $C^\perp = \bigoplus_{i=1}^{6} e_i C_i^\perp = \bigoplus_{i=1}^{6} e_i C_i = C$. Hence C is a self-dual linear code over R. The even characteristic case also follows due to the direct sum decomposition in a similar way. □

Definition 3.8. *[19] A linear code C of length nm over $\mathbb{F}_q$ is said to be a Θ_t-quasi-cyclic code of index m if $\rho_m(C) = C$, where ρ_m is the Θ_t-quasi-cyclic shift on $(\mathbb{F}_q^n)^m$ defined by*

$$\rho_m(a^1 \mid a^2 \mid \cdots \mid a^m) = (\sigma(a^1) \mid \sigma(a^2) \mid \cdots \mid \sigma(a^m)), \tag{3}$$

and σ is the Θ_t-cyclic shift operator.

Lemma 3.9. *Let σ be the Θ_t-cyclic shift. Let ρ_6 and ρ_3 be the Θ_t-quasi-cyclic shifts defined in equation (3). In the odd characteristic case, let $\phi_o : R^n \to \mathbb{F}_q^{6n}$ be the Gray map defined in equation (1). Then $\phi_o\sigma = \rho_6\phi_o$, and the even characteristic case, let $\phi_e : R^n \to \mathbb{F}_q^{3n}$ be the Gray map defined in equation (1). Then $\phi_e\sigma = \rho_3\phi_e$.*

Proof. Suppose that the characteristic is odd. Let $r_j = \sum_{i=1}^{6} e_i a_j^i \in R$ for $0 \le j \le n-1$ where $a_j^i \in \mathbb{F}_q$ for $1 \le i \le 6$. Then $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$. Now,

$$\begin{aligned}\phi_o\sigma(r) &= \phi(\Theta_t(r_{n-1}), \Theta_t(r_0), \ldots, \Theta_t(r_{n-2}))\\ &= (\theta_t(a_{n-1}^1), \theta_t(a_0^1), \ldots, \theta_t(a_{n-2}^1), \ldots, \theta_t(a_{n-1}^6), \theta_t(a_0^6), \ldots, \theta_t(a_{n-2}^6)).\end{aligned}$$

On the other hand,

$$\begin{aligned}\rho_6\phi_o(r) &= \rho_6(a_0^1, \ldots, a_{n-1}^1, \ldots, a_0^6, \ldots, a_{n-1}^6)\\ &= (\theta_t(a_{n-1}^1), \theta_t(a_0^1), \ldots, \theta_t(a_{n-2}^1), \ldots, \theta_t(a_{n-1}^6), \theta_t(a_0^6), \ldots, \theta_t(a_{n-2}^6)).\end{aligned}$$

Therefore, $\phi_o\sigma = \rho_6\phi_o$. For even characteristic the proof is similar. □

Theorem 3.10. *Let C be a linear code over R. In the odd characteristic case, C is a Θ_t-cyclic code if and only if $\phi_o(C)$ is a Θ_t- quasi-cyclic code of index 6 over $\mathbb{F}_q$, and the even characteristic case, C is a Θ_t-cyclic code if and only if $\phi_e(C)$ is a Θ_t-quasi-cyclic code of index 3 over the corresponding ring.*

Proof. Suppose that the characteristic of R is odd. Let C be a Θ_t-cyclic code of length n over R. Then $\sigma(C) = C$. By Lemma 3.9, we have $\phi_o(\sigma(C)) = \phi_o(C) = \rho_6(\phi_o(C))$. It follows that $\rho_6(\phi_o(C)) = \phi_o(C)$. Hence, $\phi_o(C)$ is a Θ_t-quasi-cyclic code of length $6n$ and index 6 over $\mathbb{F}_q$.

Conversely, suppose that $\phi_o(C)$ is a Θ_t-quasi-cyclic code of length $6n$ and index 6 over $\mathbb{F}_q$. Then $\rho_6(\phi_o(C)) = \phi_o(C)$. Again, by Lemma 3.9, we obtain $\phi_o(\sigma(C)) = \rho_6(\phi_o(C)) = \phi_o(C)$. Since ϕ_o is a bijection, it follows that $\sigma(C) = C$. Therefore, C is a Θ_t-cyclic code of length n over R.

The proof for the even characteristic case follows similarly by replacing ϕ_o with ϕ_e and ρ_6 with ρ_3. □

4. Structure of Θ_t-cyclic codes-odd characteristic case

The θ_t-cyclic codes of length n over finite field with order not dividing the length of the code are studied by Siap et al. [30]. They have shown that these codes are principally generated by the monic right divisors of $x^n - 1$ in $\mathbb{F}_q[x;\theta_t]$. Using the structure of finite fields, here first we present their properties on the ring R with odd characteristic.

Lemma 4.1. *[30] Let C be a θ_t-cyclic code of length n over $\mathbb{F}_q$ with odd characteristic. Then there exists a polynomial $f(x) \in \mathbb{F}_q[x;\theta_t]$ such that $C = \langle f(x)\rangle$ and $x^n - 1 = g(x)f(x)$ in $\mathbb{F}_q[x;\Theta_t]$.*

We recall from [9] that if $C = \langle f(x)\rangle$ is a θ_t-cyclic code of length n over $\mathbb{F}_q$ such that $x^n - 1 = g(x)f(x)$, and the order of θ_t divides n, then its dual $C^\perp = \langle g^*(x)\rangle$ is also a θ_t-cyclic code where $g^*(x)$(called it θ_t-reciprocal polynomial) is given by $g^*(x) = g_{n-r} + \theta_t(g_{n-r-1})x + \cdots + \theta_t^{n-r-1}(g_1)x^{n-r-1} + \theta_t^{n-r}(g_0)x^{n-r}$, for the polynomial $g(x) = g_0 + g_1x + \cdots + g_{n-r}x^{n-r}$.

Theorem 4.2. *Let $C = \bigoplus_{i=1}^{6} e_i C_i$ be a linear code of length n over R with odd characteristic. Then C is a Θ_t-cyclic code if and only if the codes C_i, $i = 1, 2, \dots, 6$ are θ_t-cyclic codes of length n over $\mathbb{F}_q$.*

Proof. Let $C = \bigoplus_{i=1}^{6} e_i C_i$ be a Θ_t-cyclic code of length n over R with odd characteristic. Let

$$z^i = (z_0^i, z_1^i, \dots, z_{n-1}^i) \in C_i \text{ for } 1 \le i \le 6, \text{ and } y_j = \sum_{i=1}^{6} e_i z_j^i \text{ for } 0 \le j \le n-1.$$

Then $y = (y_0, y_1, \dots, y_{n-1}) \in C$ and hence $\sigma(y) = (\Theta_t(y_{n-1}), \Theta_t(y_0), \dots, \Theta_t(y_{n-2})) \in C$. Now,

$$\sigma(y) = \sum_{i=1}^{6} e_i \sigma(z^i) \in C = \bigoplus_{i=1}^{6} e_i C_i.$$

Therefore, $\sigma(z^i) \in C_i$ for $1 \le i \le 6$. Hence $C_i's$ $(i = 1, 2, \dots, 6)$ are θ_t-cyclic code of length n over $\mathbb{F}_q$. For the converse part, let the codes C_i, $i = 1, 2, \dots, 6$ be θ_t-cyclic codes of length n over $\mathbb{F}_q$. Let $y = (y_0, y_1, \dots, y_{n-1}) \in C$, where $y_j = \sum_{i=1}^{6} e_i z_j^i$ for $0 \le j \le n-1$. Then $z^i = (z_0^i, z_1^i, \dots, z_{n-1}^i) \in C_i$ for $1 \le i \le 6$ and hence $\sigma(z^i) \in C_i$ for $1 \le i \le 6$. Again $\sigma(y) = \sum_{i=1}^{6} e_i \sigma(z^i) \in \bigoplus_{i=1}^{6} e_i C_i = C$. Consequently, C is a Θ_t-cyclic code of length n over R. □

Theorem 4.3. *Let $C = \bigoplus_{i=1}^{6} e_i C_i$ be a Θ_t-cyclic code of length n over R with odd characteristic. Then*

$$C = \langle e_1 f_1(x), e_2 f_2(x), \cdots, e_6 f_6(x) \rangle$$

and $\mid C \mid = q^{6n - \sum_{i=1}^{6} \epsilon_i}$, where $C_i = \langle f_i(x) \rangle$ and $x^n - 1 = g_i(x) f_i(x)$ in $\mathbb{F}_q[x; \theta_t]$ and $deg(f_i(x)) = \epsilon_i$, for $1 \le i \le 6$.

Proof. Let $C = \bigoplus_{i=1}^{6} e_i C_i$ be a Θ_t-cyclic code of length n over R with odd characteristic. Then by Theorem 4.2, $C_i's$ $(i = 1, 2, \dots, 6)$ are θ_t-cyclic codes of length n over $\mathbb{F}_q$. Now, by Lemma 4.1, we have $C_i = \langle f_i(x) \rangle$ and $x^n - 1 = g_i(x) f_i(x)$ in $\mathbb{F}_q[x; \theta_t]$ for $1 \le i \le 6$. Then $e_i f_i(x) \in C$ for $1 \le i \le 6$. Also for any $f(x) \in C$, we have $f(x) = \sum_{i=1}^{6} e_i h_i(x) f_i(x)$ where $h_i(x) \in \mathbb{F}_q[x; \theta_t]$ for $1 \le i \le 6$. Thus $f(x) \in \langle e_1 f_1(x), e_2 f_2(x), \cdots, e_6 f_6(x) \rangle$. Therefore, $C = \langle e_1 f_1(x), e_2 f_2(x), \cdots, e_6 f_6(x) \rangle$.
Further, we have $\mid C_i \mid = q^{n - \epsilon_i}$ and $\mid C \mid = \prod_{i=1}^{6} \mid C_i \mid = q^{6n - \sum_{i=1}^{6} \epsilon_i}$, where $deg(f_i(x)) = \epsilon_i$, for $1 \le i \le 6$. □

Corollary 4.4. *If $C = \bigoplus_{i=1}^{6} e_i C_i$ is a Θ_t-cyclic code of length n over R with odd characteristic, there exist a polynomial $f(x) \in R[x; \Theta_t]$ such that $C = \langle f(x) \rangle$ and $x^n - 1 = g(x) f(x)$ in $R[x; \Theta_t]$.*

Proof. Let $C = \bigoplus_{i=1}^{6} e_i C_i$ be a Θ_t-cyclic code of length n over R with odd characteristic. Then by Theorem 4.3, we have $C = \langle e_1 f_1(x), e_2 f_2(x), \cdots, e_6 f_6(x) \rangle$ where $C_i = \langle f_i(x) \rangle$ and $x^n - 1 = g_i(x) f_i(x)$ in $\mathbb{F}_q[x; \theta_t]$ for $1 \le i \le 6$. Let $f(x) = \sum_{i=1}^{6} e_i f_i(x) \in R[x; \Theta_t]$. Then $f(x) \in C$. On the other hand $e_i f_i(x) = e_i f(x) \in \langle f(x) \rangle$ for $i = 1, 2, \dots, 6$. Consequently, $C = \langle f(x) \rangle$. Further, $[\sum_{i=1}^{6} e_i g_i(x)] f(x) = \sum_{i=1}^{6} e_i g_i(x) f_i(x) = \sum_{i=1}^{6} e_i (x^n - 1) = x^n - 1$. Then $x^n - 1 = g(x) f(x)$ in $R[x; \Theta_t]$, where $g(x) = \sum_{i=1}^{6} e_i g_i(x)$. □

Theorem 4.5. *Let $C = \bigoplus_{i=1}^{6} e_i C_i$ be a Θ_t-cyclic code of length n over R with odd characteristic. If $gcd(n, \frac{m}{t}) = 1 = gcd(n, q)$, then there exist idempotent polynomial $i(x) \in R[x; \Theta_t]$ such that $C = \langle i(x) \rangle$ and $i(x)$ is a right divisor of $x^n - 1$.*

Proof. Let $gcd(n, \frac{m}{t}) = 1 = gcd(n, q)$. Then, there exists idempotent polynomials $i_j(x)$ (see [18]) such that $C_j = \langle i_j(x) \rangle$ and $i_j(x)$ is a right divisor of $x^n - 1$ in $\mathbb{F}_q[x; \Theta_t]$ for $1 \le j \le 6$. Then similar to Corollary 4.4, we have $C = \langle i(x) \rangle$ and $i(x)$ is a right divisor of $x^n - 1$. □

Theorem 4.6. *Let $C = \bigoplus_{i=1}^{6} e_i C_i$ be a Θ_t-cyclic code of length n over R with odd characteristic. Then $C^\perp = \bigoplus_{i=1}^{6} e_i C_i^\perp$ is also a Θ_t-cyclic code of length n over R and $\mid C^\perp \mid = q^{\sum_{i=1}^{6} \epsilon_i}$ where $C_i = \langle f_i(x) \rangle$ and $deg(f_i(x)) = \epsilon_i$ for $1 \leq i \leq 6$.*

Proof. Let $C = \bigoplus_{i=1}^{6} e_i C_i$ be a Θ_t-cyclic code of length n over R with odd characteristic. Then by Theorem 4.2, $C_i' s$ $(i = 1, 2, \ldots, 6)$ are θ_t-cyclic codes of length n over $\mathbb{F}_q$. Then $C_i^\perp$'s $(i = 1, 2, \ldots, 6)$ are also θ_t-cyclic codes of length n over $\mathbb{F}_q$. Again by Theorem 4.2, $C^\perp = \bigoplus_{i=1}^{6} e_i C_i^\perp$ is also a Θ_t-cyclic code of length n over R. Further, $\mid C_i^\perp \mid = q^{\epsilon_i}$ for $i = 1, 2, \ldots, 6$, therefore $\mid C^\perp \mid = \prod_{i=1}^{6} \mid C_i^\perp \mid = q^{\sum_{i=1}^{6} \epsilon_i}$. where $deg(f_i(x)) = \epsilon_i$ for $1 \leq i \leq 6$. □

Corollary 4.7. *Let $C = \bigoplus_{i=1}^{6} e_i C_i$ be a Θ_t-cyclic code of length n over R with odd characteristic and order of Θ_t divides n where $C_i = \langle f_i(x) \rangle$ and $x^n - 1 = g_i(x) f_i(x)$ for $1 \leq i \leq 6$. Then there exists a polynomial $G(x)$ such that $\mathcal{C}^\perp = \langle G(x) \rangle$, where $G(x) = \sum_{i=1}^{6} e_i g_i^*(x)$.*

Proof. Let $C = \bigoplus_{i=1}^{6} e_i C_i$ be a Θ_t-cyclic code of length n over R with odd characteristic and $C_i = \langle f_i(x) \rangle$ where $x^n - 1 = g_i(x) f_i(x)$ for $i = 1, 2 \ldots, 6$. Then by Theorem 4.6, we have $C^\perp = \bigoplus_{i=1}^{6} e_i C_i^\perp$ is a Θ_t-cyclic code of length n over R where $C_i^\perp$'s $(i = 1, 2, \ldots, 6)$ are θ_t-cyclic codes over $\mathbb{F}_q$. Therefore, $C_i^\perp = \langle g_i^*(x) \rangle$ where $g_i^*(x)$ is the θ_t-reciprocal polynomial of $g_i(x)$ for $1 \leq i \leq 6$. Take $G(x) = \sum_{i=1}^{6} e_i g_i^*(x)$, then we check that $C^\perp = \langle G(x) \rangle$. □

Remark 4.8. *Note that from Corollary 4.4 and Corollary 4.7 we can conclude that every Θ_t-cyclic code and their dual are principally generated left $R[x; \Theta_t]$-submodules of R_n.*

Example 4.9. *Let $q = 27$ and $R = \mathbb{F}_{27}[u, v, w]/\langle u^2 = u, v^2 = v, w^2 = 1, uv = vu = 0, uw = wu, wv = vw \rangle$. Let α be a root of an irreducible primitive polynomial over $\mathbb{F}_3$. Now define the automorphism $\theta_1(a) = a^3$ for all $a \in \mathbb{F}_{27}$ and Let $r = \sum_{i=1}^{6} e_i r_i$, where $r_i \in \mathbb{F}_{27}$ for $1 \leq i \leq 6$. Then $\Theta_1(r) = \sum_{i=1}^{6} e_i \theta_1(r_i) = \sum_{i=1}^{6} e_i r_i^3$. Let C be a Θ_1-cyclic code of length 6 over R. Again we have*

$$\begin{aligned} x^6 - 1 &= (x+1)(x+\alpha)(x+\alpha^5)(x+\alpha^7)(x+\alpha^8)(x+\alpha^{18}) \in \mathbb{F}_{27}[x; \theta_1] \\ &= (x+\alpha^2)(x+\alpha^8)(x^2+\alpha^{16})(x+\alpha^{19})(x+\alpha^{23})^2 \in \mathbb{F}_{27}[x; \theta_1] \\ &= (x+1)^3(x+2)(x+\alpha^5)(x+\alpha^{21}) \in \mathbb{F}_{27}[x; \theta_1]. \end{aligned}$$

Let $f_1(x) = f_2(x) = (x+\alpha^8)(x+\alpha^{18}), f_3(x) = f_4(x) = (x+\alpha^{19})(x+\alpha^{23})^2$ and $f_5(x) = f_6(x) = (x+\alpha^5)(x+\alpha^{21})$. Then $C = \langle 2(1-\alpha)(u+v)(x+\alpha^8)(x+\alpha^{18}) + 2(1+\alpha)(u+v)(x+\alpha^{19})(x+\alpha^{23})^2 + (1-u-v)(x+\alpha^5)(x+\alpha^{21}) \rangle$ is a Θ_1-cyclic code of length 6 over R. Further, the Gray image $\phi_o(C)$ is a $[36, 22, 3]$ linear code over the field $\mathbb{F}_{27}$.

5. Structure of Θ_t-cyclic codes-even characteristic case

Now we study the structure of Θ_t-cyclic codes in case of an even characteristic. The structure of ideals in $R_{w,x} = R_w[x; \theta_t]/\langle x^n - 1 \rangle$ depends on the factorization of $x^n - 1$. Recall that the order of Θ_t divides n and $\bar{w} = 1 + w$ and clearly $\bar{w}^2 = 0$. First we establish the following useful observation. Suppose that

$$(f_1(x) + f_2(x)\bar{w})(h_1(x) + h_2(x)\bar{w}) = x^n - 1$$

$$f_1(x)h_1(x) + (f_2(x)h_1(x) + f_1(x)h_2(x))\bar{w} = x^n - 1$$

This implies

$$f_1(x)h_1(x) = x^n - 1 \tag{4}$$

and

$$f_2(x)h_1(x) + f_1(x)h_2(x) = 0 \quad (5)$$

$\in R_w[x;\theta_t]$.

Focusing on Equation (5),

$$f_2(x)h_1(x) + f_1(x)h_2(x) = 0 \text{ implies } h_1(x)f_2(x)h_1(x) + h_1(x)f_1(x)h_2(x) = 0$$

By Equation (4), we have $f_1(x)h_1(x) = x^n - 1$. Since $x^n - 1$ is central in the skew polynomial ring, it follows that $f_1(x)$ and $h_1(x)$ commute. Hence, we have

$$h_1(x)f_2(x)h_1(x) + f_1(x)h_1(x)h_2(x) = 0 \text{ implying } h_1(x)f_2(x)h_1(x) + (x^n - 1)h_2(x) = 0$$

$$h_1(x)f_2(x)h_1(x) + h_2(x)(x^n - 1) = 0 \text{ implying } h_1(x)f_2(x)h_1(x) + h_2(x)f_1(x)h_1(x) = 0$$

implies

$$h_1(x)f_2(x) + h_2(x)f_1(x) = 0$$

Thus, we have

$$h_1(x)f_2(x) = h_2(x)f_1(x).$$

Hence,

$$f_2(x)h_1(x) = f_1(x)h_2(x) \text{ and } h_1(x)f_2(x) = h_2(x)f_1(x).$$

Therefore, if

$$(f_1(x) + f_2(x)\bar{w})(h_1(x) + h_2(x)\bar{w}) = x^n - 1,$$

then Equations (4) and (5) must hold.

Example 5.1. *Consider a factorization of $x^4 - 1$ over $\mathbb{F}_4[w]/\langle w^2 - 1\rangle[x;\theta]$ with $\theta(\beta) = \beta^2$ for $\beta \in \mathbb{F}_4$ and $\mathbb{F}_4 = \{a + b\alpha | \alpha^2 + \alpha + 1 = 0\}$ which is*

$$x^4 - 1 = (x^2 + wx + \alpha^2)(x^2 + wx + \alpha).$$

Now, it can easily be checked the Equations (4) and (5) hold.

Example 5.2. *Let us consider factorization of $x^6 - 1$ in $F_4[w]/\langle w^2 - 1\rangle[x;\theta]$ where $\theta(a) = a^4$. Examples of factorizations*

$$x^6 - 1 = (x^3 + \alpha x^2 + \alpha^2 x + 1)(x^3 + \alpha^2 x^2 + \alpha^2 x + 1), \quad (6)$$

$$x^6 - 1 = (x^3 + \alpha^2 wx^2 + (w + \alpha^2)x + 1)(x^3 + \alpha wx^2 + (w + \alpha^2)x + 1), \quad (7)$$

and

$$x^6 - 1 = (x^3 + (\alpha w + 1)x^2 + (\alpha w + 1)x + 1)(x^3 + (\alpha^2 w + 1)x^2 + (\alpha w + 1)x + 1) \quad (8)$$

where $\alpha + 1 = \bar{\alpha}$

The following theorem presents the structure of ideals (codes) over the skew polynomial ring $\mathbb{F}_w[x;\theta]/\langle x^n - 1\rangle$ where $x^n - 1$ lies in the center of the skew ring.

The first part of the following theorem is presented in [13]. Here we restate it and further we show that an additional condition can be imposed into generators. Moreover, we state the spanning sets and determine the size of the code.

Theorem 5.3. *Assume that C is a left ideal in the ring $R_w[x;\theta]/\langle x^n-1\rangle$ where $R_w = F_{2^m}[w]/\langle w^2-1\rangle$. Then, $C = \langle f(x), \bar{w}g(x)\rangle$ $f(x) = f_0(x) + \bar{w}f_1(x)$ with $f_0(x), f_1(x), g(x) \in F_q[x,\theta]$ and $\deg(f_1(x)) < \deg(f_0(x))$ and $\deg(f_1(x)) < \deg(g(x))$ and moreover $g(x)|_r f_0(x)|_r x^n - 1$ in $F_q[x,\theta]$ where $|_r$ stands for divisibility from the right.*

Moreover, the R_w- spanning set is $S_1 \cup S_2$ where

$$S_1 = \{f(x), xf(x), \ldots, x^{n-\deg(f(x))-1}f(x)\}, \text{ and}$$

$$S_2 = \{\bar{w}g(x), \bar{w}xg(x), \ldots, \bar{w}x^{\deg(g(x))-\deg(h(x))-1}g(x)\},$$

and the size of the code $|C| = 2^{m(n-\deg(f(x)))}2^{(m-1)(\deg(g(x))-\deg(h(x)))}$.

Proof. Let C be an ideal of $R_{w,x} = R_w[\Theta_t;x]/\langle x^n-1\rangle$. It is well known that right (and left) division algorithms hold in $F_{2^m}[\Theta_t;x]$ if the divisor polynomial has a unit leading coefficient. An ideal C in $R_{w,x}$ is finitely generated. The generators in C are of two types: either $f(x) = f_1(x) + f_2(x)\bar{w}$ where $f(x)$ or $g(x)\bar{w}$ and further they are right divisors of x^n-1. If the ideal does not contain a polynomial with leading coefficient being a unit, then all elements are multiples of $\bar{w}$. In this particular case we have $C = \bar{w}D$ where D is an ideal in $F_q[x;\Theta_t]/\langle x^n-1\rangle$ and it is well known that $D = \langle g(x)\rangle$ for some $g(x)$ which is a right divisor of x^n-1. And hence, $C = \langle \bar{w}g(x)\rangle$. Otherwise, there will be at least one element with leading coefficient being a unit inside the ideal. If $f_1(x) \neq 0$ then the collection of such polynomials will generate an ideal, say $C_0 = \langle f(x) = f_0(x) + \bar{w}f_1(x)\rangle$ and we can assume that $f(x) = f_0(x) + \bar{w}f_1(x)$ is its generator and naturally it is a right divisor of x^n-1. On the other hand the collection of all multiples of the zero divisor $\bar{w}$, say the set $\bar{w}C$, will also give an ideal and we may assume that $\bar{w}C = \langle \bar{w}g(x)\rangle$. So given any element in C it is an R_w-sum of elements from C_0 and $\bar{w}C$. Thus, it is clear that $C = \langle f(x) = f_1(x) + f_2(x)\bar{w}, \bar{w}g(x)\rangle$ where $f(x), g(x)|_r x^n-1$. Further, $\bar{w}f(x) = \bar{w}f_0(x) \in C$ and $\bar{w}g(x)$. Without loss of generality we may assume that $\deg(f_1(x)) < \deg(f_0(x))$ and also $\bar{w}C = \langle g(x)\rangle$ and hence $\bar{w}f_0(x) \in \bar{w}C$ and thus $g(x)|_r f_0(x)$. And finally, by making use of the smallest degree of $g(x)$ we can use $g(x)$ to right divide $f_1(x)$ and clearly impose that $\deg(f_1(x)) < \deg(g(x))$.

Since $h(x)f(x) = x^n-1$ it is clear that the set S_1 is R_w-linearly independent. Further since $g(x)|f_0(x)$ the set S_2 is also R_w-linearly independent. If a left multiple of $f(x)$, is given then the elements of S_1 can produce it. On the other hand if a multiple of $\bar{w}$ is given, say $\bar{w}b(x)$ then by minimality og $g(x)$ we have $\bar{w}b(x) = \bar{w}q(x)g(x)$ for some $q(x) \in F_q[x;\theta]$. Since $g(x)|_r f_0(x)$ we have $f_0(x) = m(x)g(x)$. Now, by applying right division algorithm, we have $\bar{w}q(x) = \bar{w}(q_0(x)m(x) + r(x))$ with $\deg(r(x)) < \deg(m(x))$. hence, $\bar{w}b(x) = \bar{w}(q_0(x)m(x)+r(x))g(x) = \bar{w}q_0(x)m(x)g(x)+\bar{w}r(x)g(x)$. $\bar{w}q_0(x)m(x)g(x)$ and $\bar{w}r(x)g(x)$ belong to the spans of S_1 and S_2 respectively. Thus, $S_1 \cup S_2$ C as an R_w- module. Therefore, by the property of $S_1 \cup S_2$ the size of the code is also directly obtained as stated. □

Example 5.4. *Let $f(x) = x^3 + x^2 + x + 1$, $g(x) = \alpha^2 w + \alpha w$. Here, $f(x) = (\alpha w x^2 + wx + \alpha^2 w)g(x)$. Then, the code $C = \langle f(x), \bar{w}g(x)\rangle$ is a skew cyclic code of length 4. Moreover, the weight enumerator of C is $W(y) = 1 + 18y^2 + 24y^3 + 213y^4$ where the exponents are the weights and their corresponding coefficients are the number such codewords. Clearly the minimum distance of this code is 2.*

If $g(x) = 0$, then $C = \langle f_0(x) + f_1(x)\bar{w}\rangle$ with $\deg(f_1(x)) < \deg(f_0(x))$ and $f_0(x)$ being a right divisor of x^n-1 and also $f_1(x)$ may or may not be zero. In case of $g(x) \neq 0$ but $f_0(x) = 0$, then $C = \langle g(x)\bar{w}\rangle$ where $g(x)$ is a right divisor of x^n-1.

The decomposition of the ring R naturally extends over the polynomial rings such as follows

$$R[x;\Theta_t]/\langle x^n-1\rangle = R_{x,\Theta_t}u \oplus R_{x,\Theta_t}v \oplus R_{x,\Theta}(1+u+v)$$

where $R_{x,\Theta_t} = R_w[x;\Theta_t]/\langle x^n-1\rangle$.

By putting together the findings established above we can state the following theorem.

Theorem 5.5. *Let E be a Θ_t-cyclic code over R of even characteristic. Then, E is a direct sum of ideals E_i where $E_i = \langle f_i(x), \bar{w}g_i(x)\rangle$ with $i = 1,2,3$ and $f_i(x) = f_{0,i}(x) + \bar{w}f_{1,i}(x)$ and $g_i(x)$'s satisfy the properties established by Theorem 5.3.*

6. Structure of (Θ_t, λ)-cyclic codes

In the present section, we assume only the odd characteristic case and we extend our study from Θ_t-cyclic to (Θ_t, λ)-cyclic codes of length n over R. The complete structures of these codes are obtained by the decomposition method.

Definition 6.1. *[21] Let C be a linear code of length n over R and $\lambda \in R_{inv}$ be a unit in R. R_{inv} represents the invariant elements of R under the map Θ_t where $R_{inv} = \{\alpha \in R | \Theta_t(\alpha) = \alpha\} = \mathbb{F}_{p^t} + u\mathbb{F}_{p^t} + v\mathbb{F}_{p^t} + w\mathbb{F}_{p^t} + uw\mathbb{F}_{p^t} + vw\mathbb{F}_{p^t}$. Then, C is said to be a (Θ_t, λ)-cyclic code if for $c = (c_0, c_1, \ldots, c_{n-1}) \in C$, we have $\sigma_\lambda(c) = (\lambda\Theta_t(c_{n-1}), \Theta_t(c_0), \ldots, \Theta_t(c_{n-2})) \in C$. Note that for $\lambda = 1$, C is a Θ_t-cyclic code and σ_λ is the (Θ_t, λ)-cyclic shift operator.*

Theorem 6.2. *Let $R_{n,\lambda} = R[x; \Theta_t]/\langle x^n - \lambda\rangle$. A linear code C of length n over R is (Θ_t, λ)-cyclic code if and only if C is a left $R[x; \Theta_t]$-submodule of $R_{n,\lambda}$.*

Proof. Same as the proof of Theorem 2.3. □

Theorem 6.3. *Let n be an integer such that $\lambda^{n+1} = 1$. Then the map $\Gamma : R_n \longrightarrow R_{n,\lambda}$ defined by $\Gamma(a(x)) = a(\lambda x)$, is a left $R[x; \Theta_t]$-module isomorphism.*

Proof. Let $a(x), b(x) \in R_n$, where $R_n = R[x; \Theta_t]/\langle x^n - 1\rangle$ such that $a(x) = b(x)$. Thus $a(x) - b(x) \equiv 0 \ (mod\ x^n - 1)$. Replacing x by λx on both sides, we have $a(\lambda x) - b(\lambda x) \equiv 0 \ (mod\ \lambda^n x^n - 1)$, i.e., $a(\lambda x) - b(\lambda x) \equiv 0,\ \lambda^n (mod\ x^n - \lambda)$. Thus $a(\lambda x) = b(\lambda x)$ in $R_{n,\lambda}$. Therefore $\Gamma(a(x)) = \Gamma(b(x))$. Consequently, Γ is an injective and well-defined map. Further, Γ is a surjective left $R[x; \Theta_t]$-module homomorphism, since for any $a(x)$, there exists a preimage $a(\lambda^{-1}x)$. Hence, the result follows.

□

Corollary 6.4. *Let n be an integer such that $\lambda^{n+1} = 1$. If C is a Θ_t-cyclic code of length n over R, then $\Gamma(C)$ is a (Θ_t, λ)-cyclic code of length n over R.*

Proof. Let C be a Θ_t-cyclic code of length n over R. In other words, C is a left $R[x; \Theta_t]$-submodule of R_n. By Theorem 6.3, $\Gamma(C)$ is a left $R[x; \Theta_t]$-submodule of $R_{n,\lambda}$. Thus $\Gamma(C)$ is a (Θ_t, λ)-cyclic code of length n over R. □

Lemma 6.5. *[16] Let C be a (θ_t, α)-cyclic code of length n over $\mathbb{F}_q$. Then there exists a polynomial $f(x) \in \mathbb{F}_q[x; \theta_t]$ such that $C = \langle f(x)\rangle$ and $x^n - \alpha = g(x)f(x)$ in $\mathbb{F}_q[x; \theta_t]$.*

Lemma 6.6. *Let $\lambda \in R$ be a non-zero element such that $\lambda = \sum_{i=1}^{6} e_i\lambda_i$, where $\lambda_i \in \mathbb{F}_q$ for $1 \le i \le 6$. Then λ is a unit in R if and only if the elements $\lambda_i,\ i = 1, 2, \ldots, 6$ are units in $\mathbb{F}_q$.*

Proof. Let $\lambda = \sum_{i=1}^{6} e_i\lambda_i$ be a unit in R where $\lambda_i \in \mathbb{F}_q$ for $1 \le i \le 6$. Then there exists a unit $\lambda' = \sum_{i=1}^{6} e_i\lambda'_i$ where $\lambda'_i \in \mathbb{F}_q^*$ for $1 \le i \le 6$. Now $\lambda\lambda' = 1$ implies $\sum_{i=1}^{6} e_i\lambda_i\lambda'_i = 1$, i.e., $e_i\lambda_i\lambda'_i = e_i$, and hence $\lambda_i\lambda'_i = 1$ for $1 \le i \le 6$. Therefore, $\lambda'_i s\ (i = 1, 2, \ldots, 6)$ are units in $\mathbb{F}_q$.
For converse part, let the elements $\lambda_i,\ i = 1, 2, \ldots, 6$ be units in $\mathbb{F}_q$. Then $\lambda\lambda' = 1$ where $\lambda' = \sum_{i=1}^{6} e_i\lambda_i^{-1} \in R$. Therefore, λ is a unit in R. □

Here, we study the (θ_t, λ)-cyclic codes of length n over R where $\lambda = a_1 + a_2u + a_3v + a_4w + a_5uw + a_6vw \in R$ is a unit and $a_i \in \mathbb{F}_q$ for $1 \le i \le 6$. By calculation the units, $\lambda'_i s$ are given by

$$\begin{aligned} &\lambda_1 = a_1 + a_2 - a_4 - a_5, \lambda_2 = a_1 + a_3 - a_4 - a_6,\\ &\lambda_3 = a_1 + a_2 + a_4 + a_5, \lambda_4 = a_1 + a_3 + a_4 + a_6,\\ &\lambda_5 = a_1 - a_4, \lambda_6 = a_1 + a_4. \end{aligned} \tag{9}$$

Theorem 6.7. *Let $C = \bigoplus_{i=1}^{6} e_i C_i$ be a linear code of length n over R. Then C is a (Θ_t, λ)-cyclic code if and only if the codes C_i are (θ_t, λ_i)-cyclic codes over $\mathbb{F}_q$, respectively for $1 \leq i \leq 6$, where $\lambda_i's$ are given by equation (9).*

Proof. Let C be a (Θ_t, λ)-cyclic code of length n over R. Let $z^i = (z_0^i, z_1{}^i, \ldots, z_{n-1}^i) \in C_i$ for $1 \leq i \leq 6$ and $y_j = \sum_{i=1}^{6} e_i z_j^i$ for $0 \leq j \leq n-1$. Now $y = (y_0, y_1, \ldots, y_{n-1}) \in C$ and hence $\sigma_\lambda(y) = (\lambda\Theta_t(y_{n-1}), \Theta_t(y_0), \ldots, \Theta_t(y_{n-2})) \in C$. Again we have $\sigma_\lambda(y) = \sum_{i=1}^{6} e_i \sigma_{\lambda_i}(z^i) \in C = \bigoplus_{i=1}^{6} e_i C_i$. Therefore, $\sigma_{\lambda_i}(z^i) \in C_i$ for $1 \leq i \leq 6$. Hence $C_i's$ are (θ_t, λ_i)-cyclic codes over $\mathbb{F}_q$, respectively for $1 \leq i \leq 6$. On the other side, let $C_i's$ be (θ_t, λ_i)-cyclic codes over $\mathbb{F}_q$, respectively for $1 \leq i \leq 6$. Let $y = (y_0, y_1, \ldots, y_{n-1}) \in C$ where $y_j = \sum_{i=1}^{6} e_i z_j^i$ for $0 \leq j \leq n-1$. Then $z^i = (z_0^i, z_1^i, \ldots, z_{n-1}^i) \in C_i$ for $1 \leq i \leq 6$ and hence $\sigma_{\lambda_i}(z^i) \in C_i$ for $1 \leq i \leq 6$. Now $\sigma_\lambda(y) = \sum_{i=1}^{6} e_i \sigma_{\lambda_i}(z^i) \in \bigoplus_{i=1}^{6} e_i C_i = C$. Therefore, $C = \bigoplus_{i=1}^{6} e_i C_i$ is (Θ_t, λ)-cyclic code of length n over R. □

Theorem 6.8. *Let $C = \bigoplus_{i=1}^{6} e_i C_i$ be a (Θ_t, λ)-cyclic code of length n over R. Then*

$$C = \langle e_1 f_1(x), e_2 f_2(x), \ldots, e_6 f_6(x) \rangle$$

where $C_i = \langle f_i(x) \rangle$ and $x^n - \lambda_i = g_i(x) f_i(x)$ in $\mathbb{F}_q[x; \theta_t]$ for $1 \leq i \leq 6$.

Proof. Let $C = \bigoplus_{i=1}^{6} e_i C_i$ be a (Θ_t, λ)-cyclic code of length n over R. Then by Theorem 6.7, the codes C_i are (θ_t, λ_i)-cyclic codes of length n over $\mathbb{F}_q$, for $1 \leq i \leq 6$. Again by Lemma 6.5, we have $C_i = \langle f_i(x) \rangle$ and $x^n - \lambda_i = g_i(x) f_i(x)$ in $\mathbb{F}_q[x; \theta_t]$ for $1 \leq i \leq 6$. Therefore, $C = \langle e_1 f_2(x), e_2 f_2(x), \ldots, e_6 f_6(x) \rangle$. □

Corollary 6.9. *Let $C = \bigoplus_{i=1}^{6} e_i C_i$ be a (Θ_t, λ)-cyclic code of length n over R. Then there exist polynomial $f(x) \in R[x; \Theta_t]$ such that $C = \langle f(x) \rangle$ and $x^n - \lambda = g(x) f(x)$ in $R[x; \Theta_t]$.*

Proof. Let $C = \bigoplus_{i=1}^{6} e_i C_i$ be a (Θ_t, λ)-cyclic code of length n over R. Then by Theorem 6.8, we have $C = \langle e_1 f_1(x), e_2 f_2(x), \ldots, e_6 f_6(x) \rangle$, where $C_i = \langle f_i(x) \rangle$ and $x^n - \lambda_i = g_i(x) f_i(x)$ in $\mathbb{F}_q[x; \theta_t]$ for $1 \leq i \leq 6$. Take $f(x) = \sum_{i=1}^{6} e_i f_i(x)$. Then $\langle f(x) \rangle \subseteq C$. Conversely, $e_i f_i(x) = e_i f(x) \in \langle f(x) \rangle$ for $1 \leq i \leq 6$. Thus $C \subseteq \langle f(x) \rangle$. Combining both sides, we conclude $C = \langle f(x) \rangle$. Further, $[\sum_{i=1}^{6} e_i g_i(x)] f(x) = \sum_{i=1}^{6} e_i g_i(x) f_i(x) = \sum_{i=1}^{6} e_i (x^n - \lambda_i) = x^n - \lambda$. Thus $x^n - \lambda = g(x) f(x)$ in $R[x; \Theta_t]$ where $g(x) = \sum_{i=1}^{6} e_i g_i(x)$. □

Theorem 6.10. *Let $C = \bigoplus_{i=1}^{6} e_i C_i$ be a (Θ_t, λ)-cyclic code of length n over R. If $gcd(n, \frac{m}{t}) = 1 = gcd(n, q)$, then there exist idempotent polynomial $i(x) \in R[x; \Theta_t]$ such that $C = \langle i(x) \rangle$ and $i(x)$ is a right divisor of $x^n - \lambda$.*

Proof. Since $gcd(n, \frac{m}{t}) = 1 = gcd(n, q)$, there exist idempotent polynomials $i_j(x)$ such that $C_j = \langle i_j(x) \rangle$ and $i_j(x)$ is a right divisor of $x^n - \lambda_j$ in $\mathbb{F}_q[x]$ for $1 \leq j \leq 6$. Then by similar argument as of Corollary 6.9, we have $C = \langle i(x) \rangle$ and $i(x)$ is a right divisor of $x^n - \lambda$. □

Theorem 6.11. *Let $C = \bigoplus_{i=1}^{6} e_i C_i$ be a (Θ_t, λ)-cyclic code of length n over R with order of Θ_t divides n. Then $C^\perp = \bigoplus_{i=1}^{6} e_i C_i^\perp$ be a (Θ_t, λ^{-1})-cyclic code of length n over R.*

Proof. Let $C = \bigoplus_{i=1}^{6} e_i C_i$ be a (Θ_t, λ)-cyclic code of length n over R. Then by Theorem 6.7, $C_i's$ are (θ_t, λ_i)-cyclic code of length n over $\mathbb{F}_q$, respectively for $1 \leq i \leq 6$. Since $\Theta_t(\lambda) = \lambda$, then $\theta_t(\lambda_i) = \lambda_i$ for $1 \leq i \leq 6$. Therefore, $C_i^\perp$ is a $(\theta_t, \lambda_i^{-1})$-cyclic code of length n over $\mathbb{F}_q$, respectively for $1 \leq i \leq 6$. As $\lambda^{-1} = \sum_{i=1}^{6} e_i \lambda_i^{-1}$, then by Theorem 6.7, $C^\perp = \bigoplus_{i=1}^{6} e_i C_i^\perp$ is a (Θ_t, λ^{-1})-cyclic code of length n over R. □

Corollary 6.12. *Let $C = \bigoplus_{i=1}^{6} e_i C_i$ be a (Θ_t, λ)-cyclic code of length n over R. Then $C^\perp = \langle G(x) \rangle$ where $G(x) = \sum_{i=1}^{6} e_i g_i^*(x)$ and $x^n - \lambda_i^{-1} = g_i(x) f_i(x)$ in $\mathbb{F}_q[x; \theta_t]$ for $1 \leq i \leq 6$.*

Proof. Let $C = \bigoplus_{i=1}^{6} e_iC_i$ be a (Θ_t, λ)-cyclic code of length n over R. The by Theorem 6.11, $C^\perp = \bigoplus_{i=1}^{6} e_iC_i^\perp$ is a (Θ_t, λ^{-1})-cyclic code of length n over R where $C_i^\perp$ is a $(\theta_t, \lambda_i^{-1})$-cyclic code of length n over $\mathbb{F}_q$, respectively for $1 \leq i \leq 6$. Let $C_i^\perp = \langle g_i^*(x)\rangle$ where $x^n - \lambda_i^{-1} = g_i(x)f_i(x)$ and $g_i^*(x)$ is the θ_t-reciprocal polynomial of $g_i(x)$ for $1 \leq i \leq 6$. Therefore, by similar argument as of (Θ_t, λ)-cyclic code, we conclude that $C^\perp = \langle G(x)\rangle$ where $G(x) = \sum_{i=1}^{6} e_ig_i^*(x)$. □

Proposition 6.13. *Let $C = \bigoplus_{i=1}^{6} e_iC_i$ be a linear code of length n over R. Then C is a self-dual (Θ_t, λ)-cyclic code if and only if C_i is a self-dual (θ_t, λ_i)-cyclic code respectively for $1 \leq i \leq 6$. Moreover, C is a self-dual (Θ_t, λ)-cyclic code if and only if $\lambda_i^2 = 1$, for $1 \leq i \leq 6$.*

Proof. Let C be a self-dual (Θ_t, λ)-cyclic code of length n over R. Therefore, $C = C^\perp$, and hence $C_i = C_i^\perp$ for $1 \leq i \leq 6$. Thus C_i is a self-dual (θ_t, λ_i)-cyclic code respectively for $1 \leq i \leq 6$. Converse follows analogously. Further, we know that C_i is self-dual (θ_t, λ_i)-cyclic code over $\mathbb{F}_q$ if and only if $\lambda_i^2 = 1$ for $1 \leq i \leq 6$. Hence the result follows. □

Remark 6.14. *By Corollary 6.9 and Corollary 6.12 we conclude that (Θ_t, λ)-cyclic codes and their duals are principally generated left $R[x; \Theta_t]$-submodule of $R_{n,\lambda}$ and $R_{n,\lambda^{-1}}$, respectively.*

Example 6.15. *Let $q = 25$ and $R = \mathbb{F}_{25}[u, v, w]/\langle u^2 = u, v^2 = v, w^2 = 1, uv = vu = 0, uw = wu, wv = vw\rangle$. Let β be a root of an irreducible primitive polynomial over $\mathbb{F}_5$. Now the automorphism $\theta_1(a) = a^5$ for all $a \in \mathbb{F}_{25}$ and hence for $r \in R$, we have $\Theta_1(r) = \sum_{i=1}^{6} e_i\theta_1(r_i) = \sum_{i=1}^{6} e_ir_i^5$ where $r_i \in \mathbb{F}_{25}$ and $1 \leq i \leq 6$. Let $\lambda = 1 + 3u$. Then $\lambda_1 = \lambda_3 = -1, \lambda_2 = \lambda_4 = \lambda_5 = \lambda_6 = 1$. Now,*

$$x^{10} - 1 = (x + 1)^3(x + 4)^3(x + \beta^{16})^2(x + \beta^{20})^2 \in \mathbb{F}_{25}[x; \theta_1]$$

and,

$$x^{10} + 1 = (x + 2)^2(x + 3)^2(x + \beta^2)^2(x + \beta^{10})(x + \beta^{14})(x + \beta^{22})^2 \in \mathbb{F}_{25}[x; \theta_1].$$

Let $f_1(x) = f_3(x) = (x + \beta^{16})(x + \beta^{20})^2$ and $f_2(x) = f_4(x) = f_5(x) = f_6(x) = (x + \beta^{14})(x + \beta^{22})^2$. Then $C = \langle \sum_{i=1}^{6} e_if_i(x)\rangle = \langle u(x + \beta^{16})(x + \beta^{20})^2 + (1 - u)(x + \beta^{14})(x + \beta^{22})^2\rangle$ is a $(\Theta_1, 1 + 3u)$-cyclic code of length 10 over R. Since $\lambda_i^2 = 1$ for $i = 1, 2, \ldots, 6$, then $\lambda^2 = 1$. Therefore, by Proposition 6.13, C is a self-dual $(\Theta_1, 1 + 3u)$-cyclic code over R. Hence, by Corollary 3.2, $\phi(C)$ is a self-dual $[60, 42, 3]$ linear code over $\mathbb{F}_{25}$.

7. DNA applications - even characteristic case

In this section first we cover some of the related literature and basics on DNA codes.

7.1. DNA basics and codes

DNA is formed as a helix by strings of nucleotides called A (adenine),T (thymine), G (guanine) and C (cytosine). Basically they are two sequences with entries from the four alphabet set $\{A, T, G, C\}$. In this helix formation A's are always paired with T's and G's with C's and vice versa. This is known as the complementary nature of DNA. Further this pairing appears in reverse ordering match. This pairing property is know as the Watson-Crick Complement (WCC). DNA has the ability to detect and correct errors during replication, thereby avoiding misreplications. This feature of DNA has attracted coding theorists to define algebraic codes satisfying these specific properties and study their properties. The very first studies of this type were carried out over alphabets of size four [1, 28, 29]. However, the milestone of applying the WCC property of a DNA was presented by Adleman, who solved a hard (NP- complete) problem is solved [4]. Besides investigating the algebraic structures of DNA codes, as it is an important problem in general finding good parameters of such codes is also an important problem and some parallel studies in this direction were also carried out [15, 22].

After some work on four alphabet algebraic structures, researchers have moved to larger alphabet sets such as $16 = 4^2$, $64 = 4^3$, $256 = 4^4$, etc. This extension is motivated by the four-letter alphabet underlying DNA. Tuples, triples, etc. and in general n-tuples are matched with strings of DNA strings of lengths two, three, four, and n [27, 33]. And very recent studies on DNA codes over subrings of the one presented here can be found in [11] and [7].

All work mentioned above but surely not complete is done on commutative algebraic structures. Recently, the study of DNA codes has been extended to noncommutative case and mostly known as skew cyclic codes. The first study in this direction is presented in [14]. Some recent studies of DNA codes over noncommutative algebraic structures are presented in [3, 11, 26].

In this paper, we extend some recent studies on DNA codes to a specific family of skew polynomial rings. In particular, this study extends the work in [14] done over finite fields extensions of characteristic two to codes over a family of chain rings obtained via ring extensions of finite fields of characteristic two (F_w) as defined in the previous section.

In this section we propose a matching with 4^{2s}-tuples DNA strings that we use as alphabet and elements of the ring $F_w = \{a + bw \quad | a, b \in \mathbb{F}_{4^s}, w^2 = 1\}$. For the special case $s = 2$, let us consider a codeword of length 3 say $c = (1 + \alpha w, 1 + \alpha + w, \alpha + w) \in F_w^3$. The reverse of this codeword is $c^r = (\alpha + w, 1 + \alpha + w, 1 + \alpha w) \in F_w^3$. If we randomly match say $AAAC$ with $1 + \alpha w$, $CTAA$ with $1 + \alpha + w$ and $ACAA$ with $\alpha + w$, then clearly c and c^r are matched with AAACCTAAACAA and ACAACTAAAAAC respectively. As can be seen, the reverse order of the ring elements is not reflected correctly to the reverse order of DNA strings. This is known as the reversibility problem in DNA codes and it depends on the algebraic structure properties of cyclic codes. Here we use results from [14, 27] for DNA alphabet matchings and extend to the ring F_w in order to overcome this obstacle.

In [27] a matching between 4^s order fields and s DNA tuples satisfying the reversibility property is presented via an algorithm. We call this matching τ (using the same naming as in [14]) and extend it to

$$\begin{aligned} \tau_w : F_w &\to \{\text{A,T,G,C}\}^{2s} \\ a + bw &\to (\tau(a), \tau(b)). \end{aligned}$$

and this is extended to n tuples by $\psi(c_0, c_1, ..., c_{n-1}) = (\tau_w(c_0), \tau_w(c_1), \ldots, \tau_w(c_{n-1}))$ where $c_i \in F_w$, $i \in \{0, \ldots, n-1\}$. The τ_w map that clearly presents a matching between field elements and the DNA tuples is defined explicitly in [27] via an algorithm. We do not present the full matching table of τ but recall that given an element $a \in F_{4^s}$. The mapping $\tau(a)$ associates the element a^{4^s} with a DNA string of length s and referring to the table the corresponding DNA strings for a and a^{4^s} are reverses of each other. Let $\mathbb{F}_w = \mathbb{F}_{16}[w]/\langle w^2 - 1\rangle$ with $s = 2$. Let $\tau : \mathbb{F}_{16} \to \{A, T, G, C\}^4$ be a mapping. For example (consider the explicit presentation of the algorithm with Table 1 in [14]), suppose $\tau(1) = TT$ and $\tau(\alpha) = AT$. Then

$$\tau_w(1 + \alpha w) = (\tau(1), \tau(\alpha)) = TTAT.$$

Now, consider

$$\tau_w(w(1^4 + \alpha^4 w)) = \tau_w((\alpha^4 + 1^4 w)) = (\tau(\alpha^4), \tau(1^4)) = TATT.$$

As seen above, given the definition of the map τ_w, since the set is closed under the multiplication by w, then the set always includes its DNA letter inverses naturally. $c = (1 + \alpha w, 1 + \alpha + w, \alpha + w) \in F_w^3$.

$$\begin{aligned} \tau : F_{4^{2s}} &\to \{\text{A,T,G,C}\}^{2s} \\ \beta &\to (b_0, b_1, ..., b_{2s-1}). \end{aligned}$$

Here, the reverse property and the difficulty while considering over tuples matched with DNA strings instead single letter matching.

The table that presents a matching between the letter alphabet of DNA characters to the elements of finite fields of order 4^{2s} for some positive integer s is presented originally in [27] and later adapted to skew cyclic codes over finite fields [14]. Later this table is extended for use on codes over skew polynomial rings with coefficients from the same field [14].

Definition 7.1. *Let C be a code of length n over F_q. If $c^r = (c_{n-1}, c_{n-2}, \ldots, c_1, c_0) \in C$ for all $c = (c_0, c_1, \ldots, c_{n-1}) \in C$, then C is called a reversible code.*

7.2. DNA codes from the skew polynomial ring

In this subsection, given $a + bw \in F_w = \mathbb{F}_{4^{2s}}[w]/\langle w^2 - 1\rangle$, we consider the map $\Theta_t = \theta$ defined by $\theta(a + bw) = a^{4^s} + b^{4^s} w$ which is an automorphism of order 2 of F_w.

The following definition is originally presented for finite fields in [14] and we extend it in natural way as

Definition 7.2. *Let $f(x) = a_0 + a_1 x + \ldots + a_t x^t$ be a polynomial of degree t over F_w and θ be an automorphism of F_w. $f(x)$ is said to be a reversible polynomial if $a_i = a_{t-i}$ for all $i \in \{0, 1, \ldots, t\}$, and $f(x)$ is said to be a θ-reversible polynomial if $a_i = \theta(a_{t-i})$ for all $i \in \{0, 1, \ldots, t\}$.*

Lemma 7.3. *Let n be even and $f(x) \in F_w[x; \theta]/\langle x^n - 1\rangle$ with θ of order 2 and the degree of f being odd. If $f(x)$ is a θ-reversible polynomial, then*

$$(x^i f(x))^r = \begin{cases} x^{s-i} f(x), & 0 \le i \le s, \\ x^{n+s-i} f(x), & s < i < n \end{cases} \tag{10}$$

where $s = n - (\deg(f(x) + 1)$.

Proof. Since $f(x) = \sum_{i=0}^{t} f_i x^i$ is a θ-reversible polynomial we have $f_i = \theta(f_{t-i})$ for $i = 0, 1, ..., t$ where $t = \deg(f(x))$ and $f_t \neq 0$ which implies that $f_0 = \theta(f_t) \neq 0$. Then, multiplying $f(x)$ on the left by $x^{s/2}$ and considering the coefficients of $x^{s/2} f(x)$ as an n tuple we have a reversible vector of length n such as

$$0 \ldots 0 f_0 f_1 \ldots f_{t_1} | f'_{t_1} \ldots f'_0 \ldots 0$$

where $t_1 = (t-1)/2$. Basically we move the nonzeros entries to the center and make the center visible by putting an artificial divider and noting the sub indices of $f(x)$. Now, the θ-reversible polynomial of $x^s/2f(x)$ is itself by definition. Further, the θ-reversible polynomial of $x^{s/2+1} f(x)$ is $x^{s/2-1} f(x)$. To see this first the one movement to right from the center will be the same order as one movement to the left. Since $s/2 + 1$ and $s/2 - 1$ are both even or odd at the same time the primes (θ values) will not change either and hence the claim. Following the same argument, we see that the θ-reversible polynomial of $x^{s/2+i} f(x)$ is $x^{s/2-i} f(x)$ for all $0 \le i \le s/2$ which gives the first line of the Equation (10). For the second line the first case not covered above is the following case

$$f'_0 0 \ldots 0 f_0 f_1 \ldots f_{t_1} | f'_{t_1} \ldots f'_2 f'_1.$$

This instance corresponds to multiplying $f(x)$ by x^{s+1} and since s is even the primes will stay as they are. By definition its θ-reversible polynomial is

$$f_1 f_2 \ldots f_{t_1} | f'_{t_1} \ldots f'_1 f'_0 0 \ldots 0 f_0.$$

To obtain its reversible counterpart one needs to multiply $x^{s+1} f(x)$ further by x^{n-s-2}. Again the exponent is $n - s - 2$ which is an even number and hence the primes do not change and we have the θ-reversible of the intended polynomial. This can be generalized in a similar way which gives the second line of the Equation (10) and completes the proof. □

One may think the fact that when $g(x)$ is θ-reversible then its parity check polynomial $f(x)$ is also θ-reversible which means that $f(x)g(x) = x^n - 1$. The reason for this question is naturally its relation to dual codes. But in fact the answer is negative. In Example 5.2, the factorizations in Equations (7) and (8) give examples where this property does not hold.

The following lemma is stated in [23] for the field case. Here we extend it for the quotient ring case.

Lemma 7.4. *Let $f(x) = \sum_{i=0}^{n} f_i x^i \in \mathbb{F}_q[x;\theta]$. The right division of $f(x)$ by the linear polynomial $x - \beta$ is equal to*

$$f(x) = q(x)(x - \beta) + \sum_{i=0}^{n} f_i N_i(\beta) \tag{11}$$

where $q(x) \in F_q[x;\theta]$ and $N_i(\beta) = \prod_{j=0}^{i-1} \theta^j(\beta)$.

Lemma 7.5. *This lemma is a natural extension of Lemma 7.4 to the ring $\mathbb{F}_w$. Let $f(x) = \sum_{i=0}^{n} f_i x^i \in F_w[x;\Theta_t]$ with f_n being a unit. The right division of $f(x)$ by the linear polynomial $x - \beta$ is equal to*

$$f(x) = q(x)(x - \beta) + \sum_{i=0}^{n} f_i N_i(\beta) \tag{12}$$

where $q(x) \in R_w[x;\Theta]$ and $N_i(\beta) = \prod_{j=0}^{i-1} \theta^j(\beta)$.

Proof. First recall that for $a + wb \in F_w$ $\theta_w(a + wb) = \theta(a) + w\theta(b)$ where θ is an automorphism over F_q. Also we extend the norm of an element to the elements of $\beta = a + wb \in F_w$ as

$$N_n(\beta) = \prod_{i=0}^{n-1} \theta^i(\beta)$$

, which is consistent with the notation $N_i(\beta) = \prod_{j=0}^{i-1} \theta^j(\beta)$.

As in the classical case over $\mathbb{F}_q[x;\theta]$ (see [23]) we first consider the auxiliary polynomial $x^n - N_n(\beta)$ and show that $x - \beta$ is a right divisor by applying induction on n. For $n = 1$, it is trivial. Assume it holds for n and consider

$$\begin{aligned} x^{n+1} - N_{n+1}(\beta) &= x^{n+1} - \theta_w(N_n(\beta))\beta + \theta_w(N_n(\beta))x - \theta_w(N_n(\beta))x \\ &= x^{n+1} - \theta_w(N_n(\beta))x + \theta_w(N_n(\beta))x - \theta_w(N_n(\beta))\beta \\ &= x^{n+1} - xN_n(\beta) + \theta_w(N_n(\beta))(x - \theta) \\ &= x\left(x^n - N_n(\beta)\right) + \theta_w(N_n(\beta))(x - \theta) \end{aligned}$$

and the first ot the polynomials is divisible by $x - \beta$ due to the induction hypothesis and the sum is trivially right divisible by $x - \beta$.

Now, we relate this auxiliary finding to the claim:

$$\begin{aligned} f(x) - \sum_{i=0}^{n} f_i N_i(\beta) &= \sum_{i=0}^{n} f_i x^i - \sum_{i=0}^{n} f_i N_i(\beta) \\ &= \sum_{i=0}^{n} f_i \left(x^i - N_i(\beta)\right). \end{aligned}$$

Since all the terms $x^i - N_i(\beta)$ are right divisible by $x - \beta$ so is $f(x) - \sum_{i=0}^{n} f_i N_i(\beta)$ and hence the result. □

If we take $\beta = 1$, then $\theta_w^i(1) = 1$ for all i and hence $N_i(\beta) = 1$. Thus, we have the following result as a corollary.

Corollary 7.6. *Let $f(x) = \sum_{i=0}^{n} f_i x^i \in F_w[x;\Theta_t]$ with f_n being a unit. $f(x)$ is right divisible by $x - 1$ if $\sum_{i=0}^{n} f_i = 0$.*

By making use of the definitions we have the following lemma:

Lemma 7.7. *If $f(x), g(x) \in F_{4^s}[x;\theta]$ are reversible polynomials, then so is $f(x) + g(x)\bar{w} \in F_w[x;\theta]$. Indeed, if $h_i(x) \in F_w[x;\theta]$ is a collection of reversible polynomials, then any F_w- combination of this collection is also a reversible polynomial.*

By applying the above lemma we also get the following:

Lemma 7.8. *Let $C = \langle f(x), \bar{w}g(x)\rangle$ be cyclic code (ideal) in $F_w[x;\theta]/\langle x^n - 1\rangle$ with the properties stated in Theorem 5.3 with n even, the order of θ is 2 and both deg(f) and deg(g) are odd. And further suppose that both $f(x)$ and $g(x)$ are θ-reversible polynomials. Then, the code C is a θ-reversible code.*

Now by applying Lemma 7.6 we have

Lemma 7.9. *Let $C = \langle f(x), \bar{w}g(x)\rangle$ be cyclic code (ideal) in $F_w[x;\theta]/\langle x^n - 1\rangle$ with the properties stated in Theorem 5.3 with n even, the order of θ is 2 and both deg(f) and deg(g) are odd. And further suppose that $f(x)$ is right divisible by $x - 1$, then C is a θ-complement code, i.e C contains the all one codeword.*

Now by applying the lemmas obtained above and recalling the definition of a DNA code we obtain the following theorem:

Theorem 7.10. *Let $C = \langle f(x), \bar{w}g(x)\rangle$ be a cyclic code (ideal) in $F_w[x;\theta]/\langle x^n - 1\rangle$ with the properties stated in Theorem 5.3 with n even, the order of θ is 2 and both deg(f) and deg(g) are odd. Further if both $f(x)$ and $g(x)$ are θ-reversible and $f(x)$ is right divisible by $x - 1$, then C is a DNA code.*

Now we recall a well-known theorem that is called Griesmer bound for codes over rings:

Theorem 7.11. *[31] Given a linear code C of length n minimum distance $d(C)$ over F_w where $k(C) - 1$ is defined as the rank of the minimal free F_w-submodules of F_w^n which contains C. Then,*

$$n \geq \sum_{i=0}^{k(C)-1} \lceil \frac{d(C)}{q^i} \rceil.$$

Below we present a DNA code which also optimal with respect to the Griesmer bound.

Example 7.12. *Let $g(x) = \alpha^2 x^3 + (w+\alpha)x^2 + (\alpha^2 + w)x + \alpha$ be a θ-reversible polynomial. Let $C = \langle g(x)\rangle$. And further for $f(x) = \alpha x^3 + (w+1)x + \alpha^2$ we have $f(x)g(x) = x^5 + x^4 + x^3 + x^2 + x + 1$. C contains the all one vector. So, C generates a DNA code which has reversible and complement properties as desired. Further this is a skew cyclic code of length 6, dimension 3 and minimum distance 4. By Theorem 7.11, C attains the Griesmer bound and hence it is optimal.*

Finally, given the direct sum structure of the ring studied and facts established above for each component we have the following theorem that presents the structure of DNA codes over the ring.

Theorem 7.13. *Let E be a θ-cyclic code over R_w viewed as a direct sum of codes E_i where $E_i = \langle f_i(x), \bar{w}g_i(x)\rangle$ for $i = 1, 2, 3$. If each E_i is a DNA code with reversibility and complement properties then, E is a DNA code satisfying both reversibility and complement properties.*

8. Conclusion

Since 2007 Θ_t-cyclic codes are extensively studied over different finite commutative rings especially over chain rings but there are also studies over non-chain rings which are one of the popular structures due to their complexity. Here, we determine the structure of codes over a particular family of finite non-chain rings over any characteristic. We relate linear codes over characteristic even rings to DNA codes and present their structures. As a future work, further research on DNA codes over this ring or similar rings, bound relations for distances and quantum error-correcting codes based on such rings await some attention.

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